MATH 408N PRACTICE MIDTERM 2

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Show your work for all the problems. Good luck!

(1) (a) [5 pts] Use the limit definition of the derivative to calculate $f'(x)$, if $f(x) = \frac{1}{x}$.

Solution:

By definition,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Therefore, plugging in the formula $f(x) = \frac{1}{x}$ and simplifying, we get

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}\cdot\frac{x}{x} - \frac{1}{x}\cdot\frac{x+h}{x+h}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}\cdot\frac{1}{h}}{h\cdot\frac{1}{h}} = \lim_{h\to 0} \frac{-h}{x(x+h)h} \\ &= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} \\ &= \boxed{-\frac{1}{x^2}} \end{aligned}$$

where the second-last step consists of plugging in $h = 0$.

Note: Do check that the answer you get is the same as you would get by differentiating normally!

(b) [5 pts] Use the known derivatives of $\sin(x)$ and $\cos(x)$, and whatever differentiation rules you like, to show that

$$(\sec(x))' = \sec(x)\tan(x)$$

Solution:

By definition, $\sec(x) = \frac{1}{\cos(x)}$. Thus,

$$\begin{aligned} (\sec(x))' &= \left(\frac{1}{\cos(x)}\right)' = \frac{\cos(x)\cdot(1)' - (\cos(x))'\cdot 1}{\cos(x)^2} \\ &= \frac{\cos(x)\cdot 0 - (-\sin(x))}{\cos(x)^2} = \frac{\sin(x)}{\cos(x)^2} \\ &= \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x) \end{aligned}$$

as required.

(2) Calcuate the derivatives of the following functions, using whatever rules you like. You do *not* need to simplify your answer, but it should be written only in terms of x!

(a) [5 pts] $f(x) = 5^{\sin(x)} + \tan^{-1}(x) + \ln(x)$.

Solution:

Since the derivative of a sum is the sum of the derivatives,

$$\begin{aligned} f'(x) &= (5^{\sin(x)} + \tan^{-1}(x) + \ln(x))' \\ &= (5^{\sin(x)})' + (\tan^{-1}(x))' + (\ln(x))' \end{aligned}$$

Recall that $(5^x)' = 5^x \ln(5)$ from class. Thus, using the chain rule,

$$(5^{\sin(x)})' = 5^{\sin(x)} \ln(5)(\sin(x))' = 5^{\sin(x)} \ln(5)\cos(x)$$

Therefore, plugging in the derivatives of $\tan^{-1}(x)$ and $\ln(x)$, we get

$$\boxed{f'(x) = 5^{\sin(x)} \ln(5)\cos(x) + \frac{1}{1+x^2} + \frac{1}{x}}$$

(b) [5 pts] $f(x) = \tan(x)^{x^2}$.

Solution:

Here, we're going to have to use logarithmic differentiation: we can't just use chain rule, since both the exponent and the base have an x in them. Thus, start by setting

$$y = \tan(x)^{x^2}$$

Now, take ln of both sides and apply log rules:

$$\ln(y) = \ln\left(\tan(x)^{x^2}\right) = x^2 \ln(\tan(x))$$

Now, take the derivative of both sides:

$$\begin{aligned} (\ln(y))' &= (x^2 \ln(\tan(x)))' \\ \Rightarrow \frac{1}{y}y' &= x^2(\ln(\tan(x)))' + (x^2)' \ln(\tan(x)) \\ &= x^2 \frac{1}{\tan(x)}(\tan(x))' + 2x\ln(\tan(x)) \\ &= x^2 \frac{\sec^2(x)}{\tan(x)} + 2x\ln(\tan(x)) \end{aligned}$$

where to differentiate the right-hand side, we first used the product rule, and then the chain rule to differentiate $\ln(\tan(x))$. Finally, multiplying both sides by y, we get

$$\begin{aligned} y' &= y\left(x^2 \frac{\sec^2(x)}{\tan(x)} + 2x\ln(\tan(x))\right) \\ &= \boxed{\tan(x)^{x^2}\left(x^2 \frac{\sec^2(x)}{\tan(x)} + 2x\ln(\tan(x))\right)} \end{aligned}$$

where we plug in the formula for y at the last step in order to get an answer in terms of just x.

(3) Do the following questions:

(a) [5 pts] Find y' in terms of x and y, if we have that

$$e^{xy} + y^2 + y = x$$

Solution:

Differentiating both sides with respect to x, we get

$$\begin{aligned}(e^{xy} + y^2 + y)' &= (x') \\ \Rightarrow e^{xy}(xy)' + 2yy' + y' &= 1 \\ \Rightarrow e^{xy}(xy' + y) + 2yy' + y' &= 1\end{aligned}$$

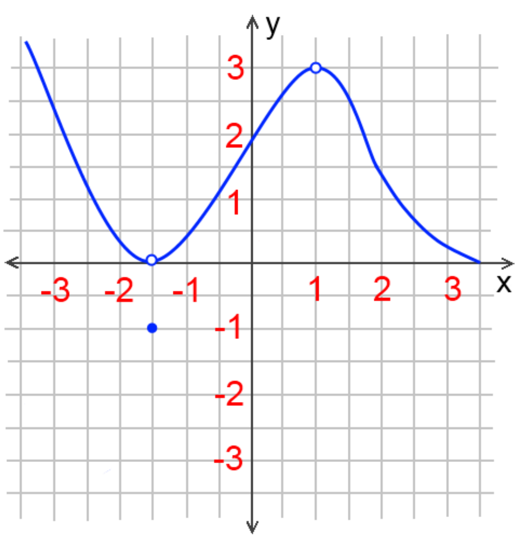
using the chain rule for e^{xy}, and then the product rule to differentiate xy.
We now need to solve for y'. We do this by first distributing everything, then putting everything with a y' on one side and everything without on the other side:

$$\begin{aligned}e^{xy}xy' + e^{xy}y + 2yy' + y' &= 1 \\ \Rightarrow e^{xy}xy' + 2yy' + y' &= 1 - e^{xy}y\end{aligned}$$

Finally, factoring out a y' from the left-hand side and dividing, we get

$$\begin{aligned}y'(e^{xy}x + 2y + 1) &= 1 - e^{xy}y \\ \Rightarrow y' &= \boxed{\frac{1 - e^{xy}y}{e^{xy}x + 2y + 1}}\end{aligned}$$

(b) [5 pts] Let $f(x)$ be given in the following picture:



Find the absolute minimum and the absolute maximum of $f(x)$ on $[-2, 2]$. If either of these doesn't exist, justify why not.

Solution:

By definition, the absolute minimum is the very smallest y-value for the x-values in $[-2, 2]$, while the absolute maximum is the largest y-value in that interval. Thus, the picture tells us that

$$\boxed{\text{Absolute minimum} = -1,\ \text{attained at } x = -1.5}$$

Now, it should be clear from looking at the graph that there is no absolute maximum – since there's a hole at the highest point between -2 and 2, whichever point we pick, we'll be able to pick another point with an even higher y-value. Thus,

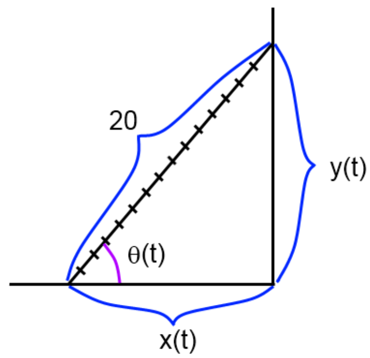
$$\boxed{\text{The absolute maximum does not exist.}}$$

(4) [10 pts] A 20 foot ladder is sliding down the wall. When the bottom of the ladder is 12 feet from the wall, the top of the ladder is sliding down at 2 ft/sec. How quickly is the angle between the ladder and the ground changing at that instant?

Solution:

Let us follow the algorithm.

1. **Draw a diagram:**



2. **Label the variables:** in the diagram above, $x(t)$ is the distance of the bottom of the ladder from the wall, $y(t)$ is the distance of the top of the ladder from the ground, and $\theta(t)$ is the angle the ladder makes with the ground.

3. **Write down information given using derivatives:** we're given that the top of the ladder is sliding down at 2 ft/sec. Writing this down with derivatives, this is precisely
$$y'(t) = -2$$
Note that the negative sign comes from the fact that the ladder is sliding down, and therefore $y(t)$ is decreasing.

4. **Write down what we want to find using derivatives:** we need to find how quickly the angle between the ladder and the ground is changing when the bottom of the ladder is 12 feet from the wall. Thus, we're looking for
$$\theta'(t) \text{ when } x(t) = 12$$

5. **Find a relationship**: since we're given $y'(t)$, and we want $\theta'(t)$, we want to relate θ and y. This relationship is precisely
$$\sin(\theta(t)) = \frac{y(t)}{20}$$

6. **Differentiate both sides of relationship with respect to t:** differentiating, (not forgetting the chain rule), we get
$$\cos(\theta(t))\theta'(t) = \frac{y'(t)}{20}$$
Now, solving for $\theta'(t)$, we get
$$\theta'(t) = \frac{y'(t)}{20\cos(\theta(t))}$$

7. **Substitute information given:** We know that $y'(t) = -2$. We also need to figure out $\cos(\theta(t))$ when $x(t) = 12$. Since cosine is opposite over adjacent, we see that when $x(t) = 12$,
$$\cos(\theta(t)) = \frac{12}{20}$$

Thus, substituting everything in,

$$\theta'(t) = \frac{-2}{20 \cdot 12/20} = -\frac{2}{12} = -\frac{1}{6}$$

Therefore, the final answer is

$$\boxed{\text{The angle is decreasing at a rate of } 1/6 \text{ radians per second.}}$$

(5) Do the following questions:

(a) [5 pts] Find the linearization of $f(x) = \sin^{-1}(x^2)$ at $x = 0$.

Solution:

The linearization is the equation of the tangent line. We will need to use the point-slope formula. With that in mind,

$$\text{Point on line} = (0, f(0)) = (0, \sin^{-1}(0)) = (0, 0)$$
$$\text{Slope of line} = f'(0)$$

Now, to find $f'(0)$, we need to find a formula for $f'(x)$. Differentiating,

$$f'(x) = (\sin^{-1}(x^2))' = \frac{1}{\sqrt{1-(x^2)^2}} \cdot (x^2)'$$
$$= \frac{2x}{\sqrt{1-x^4}}$$

Thus, plugging in, we get $f'(0) = 0$. Therefore, using point-slope, we get the equation

$$y - 0 = 0 \cdot (x - 0)$$
$$\Rightarrow y = 0$$

Therefore, the linearization is

$$\boxed{L(x) = 0}$$

(b) [5 pts] Use the result from part (a) to estimate $\sin^{-1}(0.01)$.
Useful fact: $0.01 = 0.1^2$.

Solution:

It's probably pretty clear from part (a) that this estimate isn't going to be great. However, that's what we're asked to do. We're asked to estimate $\sin^{-1}(0.01)$, which is precisely $\sin^{-1}((0.1)^2) = f(0.1)$, for $f(x) = \sin^{-1}(x^2)$. This is the function from part (a), and hence we know its linearization at $x = 0$. Therefore, we can say that

$$\sin^{-1}(0.01) = f(0.1) \approx L(0.1) = 0$$

and hence our estimate is $\boxed{0}$.