MATH 408N PRACTICE FINAL

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TA session: ____________________

Show your work for all the problems. Good luck!

(1) Calculate the following limits, using whatever tools are appropriate. State which results you're using for each question.

(a) [5 pts] $\lim_{x \to 0} \frac{e^x + 1}{x^2 + 2}$

Solution:

This one can be done straightforwardly with plugging in:

$$\lim_{x \to 0} \frac{e^x + 1}{x^2 + 2} = \frac{e^0 + 1}{0^2 + 2} = \frac{2}{2} = \boxed{1}$$

(b) [5 pts] $\lim_{x \to 0^+} x \ln(x)$

Solution:

As $x \to 0^+$, $\ln(x)$ approaches $-\infty$ and x approaches 0. Therefore, this is an indeterminate form of type $0 \cdot \infty$. Rearranging this into a fraction,

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0} \frac{\ln(x)}{1/x}$$

This is now in the form $\frac{\infty}{\infty}$. Therefore, L'Hospital's applies:

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0} \frac{\ln(x)}{1/x} = \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0} -x$$
$$= \boxed{0}$$

(c) [5 pts] $\lim_{x \to 2} \frac{\sqrt{2x} - 2}{x - 2}$

Solution:

This can also be done with L'Hospital's, since it's of the form $\frac{0}{0}$. However, let's do it with the method we learned earlier, multiplying top and bottom by the conjugate:

$$\lim_{x \to 2} \frac{\sqrt{2x} - 2}{x - 2} = \lim_{x \to 2} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2}$$
$$= \lim_{x \to 2} \frac{2x - 4}{(x - 2)(\sqrt{2x} + 2)} = \lim_{x \to 2} \frac{2}{\sqrt{2x} + 2}$$
$$= \frac{2}{\sqrt{4} + 2} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

(2) Let the function $f(x)$ be defined piecewise as follows:

$$f(x) = \begin{cases} x^2 & x < 0 \\ 1 & x = 0 \\ x & x > 0 \end{cases}$$

(a) [5 pts] Does $\lim_{x \to 0} f(x)$ exist? If yes, calculate it; if not, explain why not.

Solution:

For all these questions, it's very helpful to sketch the graph of f. I will not do so in these solutions, but I recommend it.
To check whether the limit exists, check whether the right-hand and left-hand limit match:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$$

using the fact that for positive x, $f(x) = x$ and for negative x, $f(x) = x^2$. Since the limits match,

$$\boxed{\lim_{x \to 0} f(x) = 0}$$

(b) [5 pts] Is $f(x)$ continuous at 0? Explain why or why not.

Solution:

$f(x)$ is continuous at 0 precisely if $\lim_{x \to 0} f(x) = f(0)$. Since

$$\lim_{x \to 0} f(x) = 0$$

and $f(0) = 1$, we see that $\boxed{f(x) \text{ isn't continuous at 0.}}$

(c) [5 pts] Does $f^{-1}(x)$ exist? If yes, sketch its graph; if not, explain why not.

Solution:

Using the graph of f, it is easy to see that $f(x)$ doesn't pass the horizontal line test, and hence doesn't have an inverse. If you want to do this without using a picture, note that

$$f(1) = 1 = f(-1)$$

Thus, there are two x-values for the y-value 1 – therefore, $f(x)$ doesn't have an inverse.

(3) Find the following derivatives, using whatever tools you choose. You do NOT need to simplify your answer. Unless stated otherwise, answers should only be in terms of x.

(a) [5 pts] Find $f'(x)$, where $f(x) = \dfrac{\arctan(x)}{e^x + 1}$

Solution:

Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(e^x+1)(\arctan(x))' - \arctan(x)(e^x+1)'}{(e^x+1)^2} \\ &= \boxed{\frac{(e^x+1)\cdot \frac{1}{1+x^2} - \arctan(x)e^x}{(e^x+1)^2}} \end{aligned}$$

(b) [5 pts] Find $f'(x)$, where $f(x) = \tan(x)^{2x^2}$.

Solution:

Since there is a variable in both the base and the exponent, we're forced to use logarithmic differentiation.

$$\begin{aligned} y &= \tan(x)^{2x^2} \\ \Rightarrow \ln(y) &= \ln\left(\tan(x)^{2x^2}\right) = 2x^2\ln(\tan(x)) \end{aligned}$$

Differentiating both sides with respect to x,

$$\begin{aligned} \frac{y'}{y} &= 2x^2(\ln(\tan(x)))' + (2x^2)'\ln(\tan(x)) \\ &= 2x^2\frac{1}{\tan(x)}\sec^2(x) + 4x\ln(\tan(x)) \end{aligned}$$

Finally, solving for y' and substituting in y,

$$\begin{aligned} y' &= y\left(2x^2\frac{1}{\tan(x)}\sec^2(x) + 4x\ln(\tan(x))\right) \\ &= \boxed{\tan(x)^{2x^2}\left(2x^2\frac{1}{\tan(x)}\sec^2(x) + 4x\ln(\tan(x))\right)} \end{aligned}$$

(c) [5 pts] Find y' in terms of x and y if $2^x + y = \cos(xy) + x^3$.

Solution:

This uses implicit differentiation. Differentiating both sides with respect to x,

$$\begin{aligned} 2^x\ln(2) + y' &= -\sin(xy)(xy)' + 3x^2 \\ \Rightarrow 2^x\ln(2) + y' &= -\sin(xy)(xy' + y) + 3x^2 \end{aligned}$$

Now, distributing and putting all the terms with a y' on the same side:

$$\begin{aligned} 2^x\ln(2) + y' &= -\sin(xy)xy' - \sin(xy)y + 3x^2 \\ \Rightarrow y' + \sin(xy)xy' &= -\sin(xy)y + 3x^2 - 2^x\ln(2) \end{aligned}$$

Finally, solving for y':

$$y'(1+\sin(xy)x) = -\sin(xy)y + 3x^2 - 2^x\ln(2)$$

$$\Rightarrow y' = \boxed{\frac{-\sin(xy)y + 3x^2 - 2^x\ln(2)}{1+\sin(xy)x}}$$

(4) Solve the following questions:

(a) [5 pts] Use the limit definition of the derivative to calculate $f'(x)$, where $f(x) = \frac{1}{x^2}$.

Solution:

By definition,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{x^2-(x+h)^2}{hx^2(x+h)^2} \\ &= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2} \\ &= \lim_{h\to 0} \frac{-2xh-h^2}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} \\ &= \frac{-2x}{x^2x^2} = \boxed{-\frac{2}{x^3}} \end{aligned}$$

(b) [5 pts] Find the equation of the tangent line to $f(x) = e^{x^2-1} + x$ at the point where $x = 1$.

Solution:

We need a point on the line and the slope of the line. Note that $f'(x) = 2xe^{x^2-1} + 1$

$$\text{Point} = (1, f(1)) = (1, e^0 + 1) = (1, 2)$$
$$\text{Slope} = f'(1) = 2 \cdot 1 \cdot e^0 + 1 = 3$$

Thus, using the point-slope formula,

$$\begin{aligned} &(y-2) = 3(x-1) \\ &\Rightarrow y = 3(x-1) + 2 = 3x - 3 + 2 = 3x - 1 \end{aligned}$$

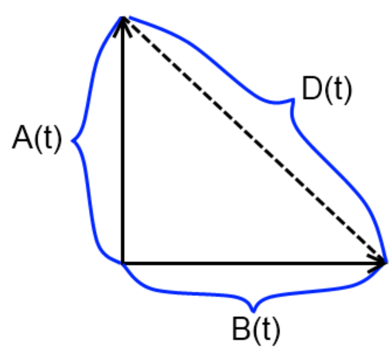
Therefore, $\boxed{y = 3x - 1}$ is the equation of the tangent line at $x = 1$.

(5) [10 pts] Two ships A and B start off at the same point, with ship A going north at the speed of 40 miles/hour, and ship B going east at the speed of 30 miles/hour. How quickly is the distance between them increasing after 2 hours?
Note: Please make sure to give descriptions for each of your variables, and to label them clearly in the picture.

Solution:

Let us follow the algorithm.

1. **Draw a diagram:**



2. **Label the variables:** in the above picture, $A(t)$ is the distance ship A has travelled after time t, and $B(t)$ is the distance ship B has travelled after time t. $D(t)$ is the distance between the ships at time t.
3. **Write down information given using derivatives:** we are given that ship A travels at 30 miles per hour, and ship B travels at 40 miles per hour. Therefore, we're given that

$$A'(t) = 40$$
$$B'(t) = 30$$

4. **Write down what we want to find using derivatives:** we're looking for how quickly the distance after two hours. Thefore, the question is:

$$\text{What is } D'(t) \text{ at } t = 2?$$

5. **Find a relationship**: it's clear that above we have a right triangle. Therefore, Pythagoras gives us that

$$A(t)^2 + B(t)^2 = D(t)^2$$

6. **Differentiate both sides of relationship with respect to t:** differentiating, (not forgetting the chain rule), we get

$$(A(t)^2 + B(t)^2)' = (D(t)^2)'$$
$$\Rightarrow 2A(t)A'(t) + 2B(t)B'(t) = 2D(t)D'(t)$$

Now, to solve for $D'(t)$ (which is our required quantity), we simply divide, getting:

$$D'(t) = \frac{2A(t)A'(t) + 2B(t)B'(t)}{2D(t)}$$
$$= \frac{A(t)A'(t) + B(t)B'(t)}{D(t)}$$

7. **Substitute information given:** we're asked about $D'(t)$ at $t = 2$. Since ship A travels at 40 miles per hour, after 2 hours, $A(t) = 2 \cdot 40 = 80$ miles, and similarly $B(t) = 2 \cdot 30 = 60$ miles. Furthermore, using the relationship between the variables, we see that after 2 hours,

$$\begin{aligned} D(t)^2 &= A(t)^2 + B(t)^2 = 80^2 + 60^2 \\ &= 6400 + 3600 = 10,000 \\ \Rightarrow D(t) &= \sqrt{10,000} = 100 \end{aligned}$$

Therefore, plugging in, we have that

$$\begin{aligned} D'(t) &= \frac{A(t)A'(t) + B(t)B'(t)}{D(t)} \\ &= \frac{80 \cdot 40 + 60 \cdot 30}{100} \\ &= \frac{3200 + 1800}{100} = \frac{5000}{100} \\ &= 50 \end{aligned}$$

Therefore, the final answer is that

$$\boxed{\text{After two hours, the distance is increasing at 50 miles per hour.}}$$

(6) (a) [5 pts] If you wanted to find an estimate for $\sqrt{4.1}$, you could use the linearization of some function $f(x)$ at some $x = a$. What should you pick $f(x)$ and a to be?

Solution:

You should clearly pick $f(x) = \sqrt{x}$, since we're estimating a square root. We should also pick $a = 4$, since that's the number closest to 4.1 with an easy square root.

(b) [5 pts] For the $f(x)$ and a you found in part (a), find the linearization $L(x)$ of $f(x)$ at a, and then use it to estimate $\sqrt{4.1}$.

Solution:

Note that $f'(x) = \frac{1}{2\sqrt{x}}$. The linearization is just the tangent line, so we use point-slope. Therefore,

$$\text{Point} = (4, f(4)) = (4, \sqrt{4}) = (4, 2)$$
$$\text{Slope} = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

Therefore, the equation of the tangent is

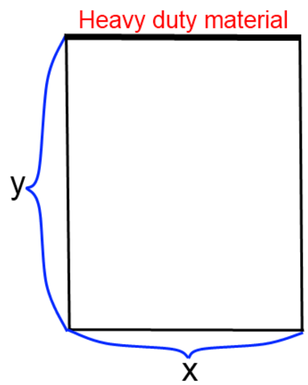
$$y - 2 = \frac{1}{4}(x - 4)$$
$$\Rightarrow y = \frac{1}{4}(x - 4) + 2 = \frac{1}{4}x + 1$$

Thus, $\boxed{L(x) = \frac{1}{4}x + 1}$.

(7) [10 pts] A rancher wishes to build a fence to enclose a rectangular pen having area 32 square yards. Along one side the fence is to be made of heavy duty material costing \$9 per yard, while the remaining three sides are to be made of cheaper material costing \$3 per yard. Determine the least cost of fencing for the pen.
Note: Please make sure to give descriptions for each of your variables, and to label them clearly in the picture. Also, don't forget to state the domain of your function, and explain which test you're using to optimize!

Solution:

1. Here's the diagram:



2. As labelled above, x is the width and y is the length of the rectangle. Furthermore, let C be the total cost of the pen.
3. The three sides not made of heavy duty material cost 3 dollars a yard – their side lengths are x, y, y. The side made of heavy duty material of length of x costs 9 dollars a yard. Thus,
$$C = 3x + 3y + 3y + 9x = 12x + 6y$$
4. Since the total area of the pen should be 32 square yards,
$$xy = 32$$
5. Here, it doesn't matter which variable to solve for. Let's solve for y:
$$xy = 32$$
$$\Rightarrow y = \frac{32}{x}$$
Thus, we get that
$$C = 12x + 6y = 12x + 6 \cdot \frac{32}{x} = 12x + \frac{192}{x}$$
6. What values of x make sense here? Obviously, we must have that $x \geq 0$ since x is a side length. Furthermore, $x = 0$ isn't consistent with having area 32 (or put another way, this leads to y being undefined.) Thus, the domain is $(0, \infty)$.
7. Here, we're maximizing
$$C(x) = 12x + \frac{192}{x}$$
over the domain $(0, \infty)$. Since we do not have a closed interval, we have to figure out where the function is increasing and decreasing. Differentiating,
$$C'(x) = 12 - \frac{192}{x^2}$$

Now, we plot points where $C'(x) = 0$ or doesn't exist on a number line. $C'(x)$ doesn't exist at 0, but that's not in our domain. Thus, set it to 0:

$$\begin{aligned} 0 &= 12 - \frac{192}{x^2} \\ \Rightarrow \frac{192}{x^2} &= 12 \\ \Rightarrow 192 &= 12x^2 \\ \Rightarrow x^2 &= \frac{192}{12} = 16 \\ \Rightarrow x &= 4 \end{aligned}$$

Therefore, the only point to be plotted is $x = 4$. We thus test the intervals $(0, 4)$ and $(4, \infty)$. Since

$$\begin{aligned} C'(1) &= 12 - \frac{192}{1^2} = -180 < 0 \\ C'(6) &= 12 - \frac{192}{6^2} = 12 - \frac{16}{3} > 0 \end{aligned}$$

we see that $C(x)$ is decreasing on $(0, 4)$ and increasing on $(4, \infty)$. Therefore, the absolute min is attained at $x = 4$, so the minimal cost is

$$C(4) = 12 \cdot 4 + \frac{192}{4} = 48 + 48 = 96$$

Therefore,

$$\boxed{\text{The minimum cost of the pen is \$96 dollars.}}$$

(8) Let $f(x) = \frac{x^2}{x^2-1}$.

(a) [5 pts] Find the intervals on which $f(x)$ is increasing and decreasing.

Solution:

This is equivalent to finding intervals on which $f'(x)$ is positive and negative. Using the quotient rule,

$$f'(x) = \frac{(x^2-1)\cdot 2x - x^2 \cdot 2x}{(x^2-1)^2} = -\frac{2x}{(x^2-1)^2}$$

To figure out where $f'(x)$ is positive and negative, first plot the points where $f'(x) = 0$ or doesn't exist on a number line.

$f'(x) = 0$ when the numerator is 0 – thus, the only possibility is $x = 0$.

$f'(x)$ doesn't exist when the denominator is 0:

$$\begin{aligned}(x^2-1)^2 &= 0\\ \Rightarrow (x^2-1) &= 0\\ \Rightarrow x^2 &= 1\\ \Rightarrow x &= \pm 1\end{aligned}$$

Therefore, $f'(x)$ could potentially change sign at $x = -1, 0, 1$. Testing, we see that $f'(x)$ is positive on $(-\infty, -1)$ and $(-1, 0)$ and negative on $(0, 1)$ and $(1, \infty)$. Thus,

$$\boxed{f(x) \text{ is increasing on } (-\infty,-1) \text{ and } (-1,0)}$$

$$\boxed{f(x) \text{ is decreasing on } (0,1) \text{ and } (1,\infty)}$$

(b) [5 pts] Find the intervals on which $f(x)$ is concave up and concave down.

Solution:

Here, we use $f''(x)$. Again using quotient rule,

$$\begin{aligned}f''(x) &= \left(-\frac{2x}{(x^2-1)^2}\right)' = -\frac{(x^2-1)^2\cdot 2 - 2x\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}\\ &= -\frac{2(x^2-1) - 8x^2}{(x^2-1)^3} = \frac{6x^2+2}{(x^2-1)^3}\end{aligned}$$

At this point, we could do the same thing as above and find places where $f''(x)$ changes sign, then test each interval. However, a little bit of thought should convince us that both the numerator and the denominator of the above fraction is always positive. Thus, $f''(x)$ is always positive except where it's not defined, which is at $x = \pm 1$. Therefore,

$$\boxed{f(x) \text{ is concave up on } (-\infty,-1),\ (-1,1) \text{ and } (1,\infty)}$$

(c) [5 pts] Find the horizontal asymptotes of $f(x)$. For each asymptote, state whether it occurs at ∞ or $-\infty$.

Solution:

To find horizontal asymptotes, take limits as $x \to \infty$ and $x \to -\infty$:

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x^2}{x^2-1} = \lim_{x\to\infty} \frac{x^2/x^2}{(x^2-1)/x^2} \\ &= \lim_{x\to\infty} \frac{1}{1-1/x^2} = \frac{1}{1-0} = \boxed{1}\end{aligned}$$

Simillarly,

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x^2}{x^2-1} = \lim_{x\to-\infty} \frac{x^2/x^2}{(x^2-1)/x^2} \\ &= \lim_{x\to-\infty} \frac{1}{1-1/x^2} = \frac{1}{1-0} = \boxed{1}\end{aligned}$$

Therefore,

$$\boxed{\text{There is the horizontal asymptote } y = 1 \text{ at } \infty.}$$

$$\boxed{\text{There is the horizontal asymptote } y = 1 \text{ at } -\infty.}$$

(d) [5 pts] Find the vertical asymptotes of $f(x)$. For each vertical asymptote $x = a$, calculate $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$.

Solution:

Vertical asymptotes correspond to places where the function goes off to ∞ and $-\infty$. This happens when the denominator is 0 or the numerator blow up. Here, the numerator doesn't blow up, so set the denominator to 0:

$$\begin{aligned}x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1\end{aligned}$$

Thus, the potential asymptotes are $x = 1$ and $x = -1$. Let us take the limits at these values from the left and right:

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{x^2}{x^2-1} \approx \frac{1.01^2}{1.01^2-1} \approx \frac{1}{\text{small positive \#}} = \text{large positive \#}$$

$$\Rightarrow \boxed{\lim_{x\to1^+} f(x) = \infty}$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{x^2}{x^2-1} \approx \frac{0.99^2}{0.99^2-1} \approx \frac{1}{\text{small negative \#}} = \text{large negative \#}$$

$$\Rightarrow \boxed{\lim_{x\to1^-} f(x) = -\infty}$$

Similarly,

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x^2}{x^2-1} \approx \frac{(-0.99)^2}{(-0.99)^2-1} \approx \frac{1}{\text{small negative \#}} = \text{large negative \#}$$

$$\Rightarrow \boxed{\lim_{x\to -1^+} f(x) = -\infty}$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^2}{x^2-1} \approx \frac{(-1.01)^2}{(-1.01)^2-1} \approx \frac{1}{\text{small positive \#}} = \text{large positive \#}$$

$$\Rightarrow \boxed{\lim_{x\to -1^-} f(x) = \infty}$$

(e) [5 pts] Use the information from the previous parts of the question to sketch the graph of $f(x)$.

(9) Let $f(x) = \frac{2x}{3} + x^{2/3}$.

(a) [5 pts] Find the critical numbers of $f(x)$.

Solution:

By definition, the critical numbers of f are the x-values in the domain of f where $f'(x) = 0$ or DNE. Here,

$$f'(x) = \frac{2}{3} + \frac{2}{3}x^{-1/3} = \frac{2}{3} + \frac{2}{3\sqrt[3]{x}} = \frac{2\sqrt[3]{x}+2}{3\sqrt[3]{x}}$$

$f'(x)$ doesn't exist if the denominator is 0. This clearly happens only if $x = 0$. Checking, $x = 0$ is in the domain of f, and hence is a critical number.
$f'(x) = 0$ if the numerator is 0:

$$\begin{aligned} 2\sqrt[3]{x} + 2 &= 0 \\ \Rightarrow 2\sqrt[3]{x} &= -1 \\ \Rightarrow \sqrt[3]{x} &= -1 \\ \Rightarrow x &= (-1)^3 = -1 \end{aligned}$$

Therefore, the critical numbers of $f(x)$ are $\boxed{x = 0 \text{ and } x = -1.}$

(b) [5 pts] Find all the local minimumus and maximums of $f(x)$.

Solution:

We need to test for intervals of increase and decrease. Plot all the points where $f'(x) = 0$ or doesn't exist on the number line. Calculations from above show that those numbers are 0 and -1. Let's test each of the intervals $(-\infty, -1)$, $(-1, 0)$ and $(0, \infty)$:

$$\begin{aligned} f'(-8) &= \frac{2}{3} + \frac{2}{3\sqrt[3]{-8}} = \frac{2}{3} + \frac{2}{3(-2)} > 0 \\ f'(-1/8) &= \frac{2}{3} + \frac{2}{3\sqrt[3]{-1/8}} = \frac{2}{3} - \frac{4}{3} < 0 \\ f'(8) &= \frac{2}{3} + \frac{2}{3\sqrt[3]{8}} = \frac{2}{3} + \frac{2}{6} > 0 \end{aligned}$$

Thus, $f(x)$ is increasing on $(-\infty, -1)$ and $(0, \infty)$, and decreasing on $(-1, 0)$. Therefore,

$$\boxed{x = -1 \text{ is a local max, } x = 0 \text{ is a local min.}}$$

(c) [5x pts] What is the absolute maximum of $f(x)$ on $[-8, 0]$? State which method you're using.

Solution:

Here, we could use the closed interval test, but this is easier to do with simple logic. Since the function is increasing on $[-8, -1)$ and decreasing on $(-1, 0]$, it's clear that the maximum is attained at $x = -1$. Thus, the absolute maximum is

$$\begin{aligned} f(-1) &= \frac{-2}{3} + (-1)^{2/3} = -\frac{1}{3} + ((-1)^2)^{1/3} \\ &= -\frac{2}{3} + 1 = \boxed{\frac{1}{3}} \end{aligned}$$

(10) Solve the following questions:

(a) [5 pts] Find the most general $F(x)$ such that $F'(x) = e^{3x} + 2x + \frac{1}{4x}$.

Solution:

Taking the antiderivative of each term, we see that the most general antiderivative is

$$\boxed{F(x) = \frac{e^{3x}}{3} + x^2 + \frac{\ln(x)}{4} + C}$$

You can check that this works by differentiating.

(b) [5 pts] Find $\int_0^{\pi/2} (\cos(2x) - 3\sin(3x))\,dx$ using whatever tools you choose.

Solution:

By the FTC, Part 2,

$$\int_0^{\pi/2} (\cos(2x) - 3\sin(3x))\,dx = F(\pi/2) - F(0)$$

where $F(x)$ is any antiderivative of $(\cos(2x) - 3\sin(3x))$. Thus, we can take

$$F(x) = \frac{\sin(2x)}{2} + \cos(3x)$$

Therefore,

$$\begin{aligned}\int_0^{\pi/2} (\cos(2x) - 3\sin(3x))\,dx &= F(\pi/2) - F(0) \\ &= \frac{\sin(\pi)}{2} + \cos(3\pi/2) - \left(\frac{\sin(0)}{2} + \cos(0)\right) \\ &= 0 + 0 - (0+1) = \boxed{-1}\end{aligned}$$

(c) [5 pts] Let $g(x) = \int_1^{\sin(x)} e^{t^2}\,dt$. Find $g'(x)$.

Solution:

This question uses Chain Rule. Let $u = \sin(x)$. Then,

$$g(x) = \int_1^u e^{t^2}\,dt$$

Using the Chain Rule,

$$\begin{aligned}g'(x) &= \left(\int_1^u e^{t^2}\,dt\right)' u'(x) \\ &= e^{u^2} u'(x) = \boxed{e^{\sin(x)^2}\cos(x)}\end{aligned}$$

using the FTC, part 1, to differentiate $\int_1^u e^{t^2}\,dt$ with respect to u.

(11) Solve the following problems:

(a) [5 pts] Estimate the area under $y = x^2 + x$ from $x = 1$ to $x = 4$ using 3 rectangles and the left endpoint rule.

Solution:

You should sketch a picture for this to see what's going on! I'm just going to write down the formulas, but they'll make more sense with a picture.
The three rectangles in questions will have the bases $[1, 2], [2, 3]$ and $[3, 4]$. Since we're using left endpoints,

$$\begin{aligned}\text{Height of first rectangle} &= f(1) = 1^2 + 1 = 2\\ \text{Height of second rectangle} &= f(2) = 2^2 + 2 = 6\\ \text{Height of third rectangle} &= f(3) = 3^2 + 3 = 12\end{aligned}$$

The base of each rectangle is clearly of length 1. Therefore, the total area of the rectangles is

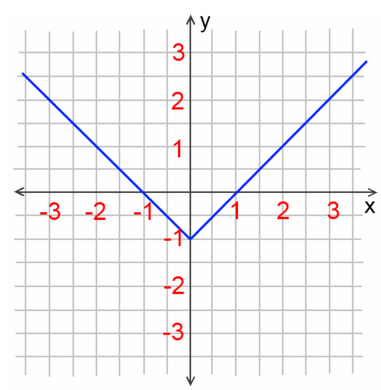
$$1 \cdot 2 + 1 \cdot 6 + 1 \cdot 12 = 19$$

and so $\boxed{\int_1^4 (x^2 + x)\, dx \approx 19}$.

(b) [5 pts] Sketch a graph of $g(x) = |x| - 1$. Use it to calculate $\int_{-1}^{3} g(x)\, dx$.

Solution:

Here's the sketch:



Using this, the integral is just the net area between $x = -1$ and $x = 3$. Thus, using the formula for the area of a triangle:

$$\int_{-1}^{3} g(x)\, dx = -\frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 2 = -1 + 2 = \boxed{1}$$

(12) (a) [5 pts] The following sum approximates a certain integral using left endpoints. What is that integral, and how many rectangles are we using? When you find your answer, please check it by actually trying to do the approximation!

$$\frac{\pi}{4}\cos(\pi) + \frac{\pi}{4}\cos\left(\frac{5\pi}{4}\right) + \frac{\pi}{4}\cos\left(\frac{6\pi}{4}\right) + \frac{\pi}{4}\cos\left(\frac{7\pi}{4}\right)$$

Solution:

This is an estimate of

$$\int_{\pi}^{2\pi} \cos(x)\, dx$$

using four rectangles. I recommend you check this answer by actually doing the estimate and seeing that you get the same sum.

(b) [5 pts] Write down the sum given in Part (a) using sigma notation.

Solution:

This sum is

$$\sum_{i=0}^{3} \frac{\pi}{4}\cos\left(\pi + \frac{i\pi}{4}\right)$$

Note that this is certainly not the only possibility! You can always check your answer by expanding out the sigma notation and seeing if you get the right thing.