Inequality Problem Solutions

1. Show that

$$1 \cdot 3 \cdot 5 \cdots (2n-1) \leq n^n$$

Solution: By the AM-GM inequality, we have that

$$\sqrt[n]{1 \cdot 3 \cdot 5 \cdots (2n-1)} \leq \frac{1+3+\cdots+(2n-1)}{n}$$

since there are n terms in $\{1, 3, \ldots, 2n-1\}$. Now, it's a well-known formula (with a beautiful visual demonstration – look it up!) that

$$1 + 3 + \cdots + (2n-1) = n^2$$

Thus, simplifying the above, we get

$$\sqrt[n]{1 \cdot 3 \cdot 5 \cdots (2n-1)} \leq \frac{n^2}{n} = n$$

Finally, taking the nth power of both sides, we get that

$$1 \cdot 3 \cdot 5 \cdots (2n-1) \leq n^n$$

as required. □

2. Show that if a, b, c are all positive, then

$$(a+b)(b+c)(a+c) \geq 8abc$$

Solution: Using AM-GM, we get that

$$\frac{a+b}{2} \geq \sqrt{ab}$$
$$\Rightarrow a + b \geq 2\sqrt{ab}$$

Similar manipulations show that

$$b + c \geq 2\sqrt{bc}$$
$$a + c \geq 2\sqrt{ac}$$

Multiplying all these inequalities together (this is allowed, since all the numbers involved are positive), we get

$$\begin{aligned}(a+b)(b+c)(a+c) &\geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ac}) \\ &= 8\sqrt{ab \cdot bc \cdot ac} = 8\sqrt{a^2b^2c^2} \\ &= 8abc\end{aligned}$$

as required. □

3. Show that if $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1+a_2+\cdots+a_n = 1$, then
$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{1}{n}$$

Solution: This question is best done using the Cauchy-Schwarz Inequality – note that it also follows from the Power Mean inequality with $r = 1$ and $s = 2$, but that inequality would only apply for $a_1, a_2, \ldots, a_n$ positive, and as such some argument would need to be made for the case where some of them are negative. Cauchy-Schwarz, on the other hand, applies to both positive and negative numbers.

Using Cauchy-Schwarz with $b_1 = b_2 = \cdots = b_n = 1$, we get that
$$(a_1^2 + a_2^2 + \cdots + a_n^2)(1^2 + 1^2 + \cdots + 1^2) \geq (a_1 \cdot 1 + a_2 \cdot 1 + \cdots + a_n \cdot 1)^2$$
$$\Rightarrow (a_1^2 + a_2^2 + \cdots + a_n^2) \cdot n \geq (a_1 + a_2 + \cdot + a_n)^2$$

Since we're given that $a_1 + a_2 + \cdots + a_n = 1$, this simplifies to
$$(a_1^2 + a_2^2 + \cdots + a_n^2) \cdot n \geq 1$$
$$\Rightarrow (a_1^2 + a_2^2 + \cdots + a_n^2) \geq \frac{1}{n}$$

as required. □

4. Show that if a, b, c are all positive, then
$$\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

Solution: From the Power Mean inequality with $r = \frac{1}{2}$ and $s = 1$, we get that
$$\left(\frac{a^{1/2} + b^{1/2} + c^{1/2}}{3}\right)^2 \leq \frac{a+b+c}{3}$$

Thus, taking the square root of both sides and rearranging, we get
$$\frac{a^{1/2} + b^{1/2} + c^{1/2}}{3} \leq \sqrt{\frac{a+b+c}{3}}$$
$$\Rightarrow \sqrt{a} + \sqrt{b} + \sqrt{c} \leq 3\sqrt{\frac{a+b+c}{3}} = \sqrt{9 \cdot \frac{a+b+c}{3}}$$
$$= \sqrt{3(a+b+c)}$$

which is precisely what we wanted. □

5. Show that
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{999999}{1000000} < \frac{1}{1000}$$

Hint: Square each side and "give a little" to create a telescoping product.

Solution: To make the calculations look a little less unwieldy, let

$$r = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{999999}{1000000}$$

Thus, what we need to show is that $r \leq \frac{1}{1000}$. Now, note that

$$r^2 = \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdots \frac{999999^2}{1000000^2}$$

Since decreasing the denominator of a fraction makes it bigger, we have that

$$\begin{aligned}
\frac{1^2}{2^2} &\leq \frac{1^2}{2^2-1} = \frac{1^2}{(2-1)(2+1)} = \frac{1^2}{1 \cdot 3} \\
\frac{3^2}{4^2} &\leq \frac{3^2}{4^2-1} = \frac{3^2}{(4-1)(4+1)} = \frac{3^2}{3 \cdot 5} \\
\frac{5^2}{6^2} &\leq \frac{5^2}{6^2-1} = \frac{5^2}{(6-1)(6+1)} = \frac{5^2}{5 \cdot 7} \\
&\vdots \qquad\qquad\qquad \vdots \\
\frac{999999^2}{1000000^2} &\leq \frac{999999^2}{1000000^2-1} = \frac{999999^2}{(1000000-1)(1000000+1)} \\
&= \frac{999999^2}{999999 \cdot 1000001}
\end{aligned}$$

Multiplying all these together, we get that

$$\begin{aligned}
r^2 &\leq \frac{1^2}{1 \cdot 3} \cdot \frac{3^2}{3 \cdot 5} \cdot \frac{5^2}{5 \cdot 7} \cdots \frac{999999^2}{999999 \cdot 1000001} \\
&= \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots 999999^2}{1 \cdot 3^2 \cdot 5^2 \cdots 999999^2 \cdot 1000001} \\
&= \frac{1}{1000001} \leq \frac{1}{1000000} \\
&= \left(\frac{1}{1000}\right)^2
\end{aligned}$$

Now, taking the square root of both sides, we get that

$$r \leq \frac{1}{1000}$$

as required. □

6. (1998 Putnam) Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

Solution: This question actually requires doing a little bit of clean up before working on he inequality. Note that

$$\begin{aligned}\left(x^3 + 1/x^3\right)^2 &= (x^3)^2 + 2 \cdot x^3 \cdot 1/x^3 + \left(1/x^3\right)^2 \\ &= x^6 + 2 + 1/x^6\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{(x+1/x)^6 - (x^6+1/x^6) - 2}{(x+1/x)^3 + (x^3+1/x^3)} &= \frac{(x+1/x)^6 - (x^6+1/x^6+2)}{(x+1/x)^3 + (x^3+1/x^3)} \\ &= \frac{(x+1/x)^6 - (x^3+1/x^3)^2}{(x+1/x)^3 + (x^3+1/x^3)}\end{aligned}$$

Now, using difference of squares, the above is precisely

$$\frac{((x+1/x)^3 + (x^3+1/x^3))((x+1/x)^3 - (x^3+1/x^3))}{(x+1/x)^3 + (x^3+1/x^3)}$$

which simplifies to

$$\begin{aligned}(x+1/x)^3 - (x^3+1/x^3) &= x^3 + 3x + 3/x + 1/x^3 - x^3 - 1/x^3 \\ &= 3x + 3/x = 3(x+1/x)\end{aligned}$$

Finally, using AM-GM, we have that

$$3(x+1/x) \geq 3 \cdot 2\sqrt{x \cdot 1/x} = 6$$

Therefore, we have gotten a lower bound of 6. Since AM-GM attains equality precisely when all the values are equal, we can have equality if $x = 1/x$. Thus, equality will be attained when $x = 1$. Trying it, we see that

$$\begin{aligned}\frac{(1+1/1)^6 - (1^6+1/1^6) - 2}{(1+1/1)^3 + (1^3+1/1^3)} &= \frac{2^6 - 2 - 2}{2^3 + 2} \\ &= \frac{64-4}{8+2} = \frac{60}{10} \\ &= 6\end{aligned}$$

So indeed, we can attain the value 6. □

7. Show that for any integer n,

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$$

Note: You may recognize these expressions: they approach e as $n \to \infty$.

Solution: Let us use AM-GM with

$$a_1 = 1, a_2 = 1 + \frac{1}{n}, \cdots, a_{n+1} = 1 + \frac{1}{n},$$

The inequality gives us that

$$\sqrt[n+1]{a_1 \cdot a_2 \cdots a_{n+1}} \leq \frac{a_1 + \cdots + a_{n+1}}{n+1}$$

Now, note that

$$\begin{aligned} a_1 \cdot a_2 \cdots a_{n+1} &= 1 \cdot \left(1 + \frac{1}{n}\right) \cdots \left(1 + \frac{1}{n}\right) \\ &= \left(1 + \frac{1}{n}\right)^n \end{aligned}$$

Also,

$$\begin{aligned} \frac{a_1 + \cdots + a_{n+1}}{n+1} &= \frac{1}{n+1}\left(1 + \left(1 + \frac{1}{n}\right) + \cdots + \left(1 + \frac{1}{n}\right)\right) \\ &= \frac{1}{n+1}\left(n + 1 + n \cdot \frac{1}{n}\right) \\ &= 1 + \frac{1}{n+1} \end{aligned}$$

Thus, plugging these back into the inequality yields

$$\sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n} \leq 1 + \frac{1}{n+1}$$

Finally, taking the $n + 1$st power of both sides gives

$$\left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n+1}\right)^{n+1}$$

as required. □

8. If a, b and c are sides of a triangle, show that

$$\frac{a}{b + c - a} + \frac{b}{a + c - b} + \frac{c}{a + b - c} \geq 3$$

Solution: It is well known that if a, b and c are sides of a triangle, then the triangle inequality tells us that

$$a + b \geq c, \ a + c \geq b, \ b + c \geq a$$

The trick to this questions is to rephrase it in terms of the quantities

$$\begin{aligned} x &= a + b - c \\ y &= a + c - b \\ z &= b + c - a \end{aligned}$$

which the triangle inequality tells us are positive. It's straightforward to check that

$$a = \frac{x+y}{2}, b = \frac{x+z}{2}, c = \frac{y+z}{2}$$

Rewriting the desired inequality in terms of x, y and z, we need to show that

$$\frac{x+y}{2z} + \frac{x+z}{2y} + \frac{y+z}{2x} \geq 3$$

for $x, y, z > 0$. This turns out to be easy to show. Using AM-GM,

$$\begin{aligned} \frac{x+y}{2z} + \frac{x+z}{2y} + \frac{y+z}{2x} &= \frac{x}{2z} + \frac{y}{2z} + \frac{x}{2y} + \frac{z}{2y} + \frac{y}{2x} + \frac{z}{2x} \\ &\geq 6\sqrt[6]{\frac{x}{2z} \cdot \frac{y}{2z} \cdot \frac{x}{2y} \cdot \frac{z}{2y} \cdot \frac{y}{2x} \cdot \frac{z}{2x}} \\ &= 6\sqrt[6]{\frac{x^2y^2z^2}{2^6x^2y^2z^2}} = 6\sqrt[6]{\frac{1}{2^6}} = 6 \cdot \frac{1}{2} \\ &= 3 \end{aligned}$$

as required. □

9. (2003 Putnam) Show that if $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then

$$(a_1a_2 \ldots a_n)^{1/n} + (b_1b_2 \ldots b_n)^{1/n} \leq [(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)]^{1/n}$$

Solution: The desired inequality is clearly equivalent to

$$\frac{(a_1a_2 \ldots a_n)^{1/n} + (b_1b_2 \ldots b_n)^{1/n}}{[(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)]^{1/n}} \leq 1$$

Using AM-GM,

$$\begin{aligned} \frac{(a_1a_2 \ldots a_n)^{1/n}}{[(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)]^{1/n}} &= \left(\frac{a_1}{a_1+b_1} \cdot \frac{a_2}{a_2+b_2} \cdots \frac{a_n}{a_n+b_n}\right)^{1/n} \\ &\leq \frac{1}{n}\left(\frac{a_1}{a_1+b_1} + \cdots + \frac{a_n}{a_n+b_n}\right) \end{aligned}$$

Similarly,

$$\frac{(b_1 b_2 \dots b_n)^{1/n}}{[(a_1+b_1)(a_2+b_2)\dots(a_n+b_n)]^{1/n}} \le \frac{1}{n}\left(\frac{b_1}{a_1+b_1} + \dots + \frac{b_n}{a_n+b_n}\right)$$

Therefore, adding these up,

$$\begin{aligned}
&\frac{(a_1 a_2 \dots a_n)^{1/n} + (b_1 b_2 \dots b_n)^{1/n}}{[(a_1+b_1)(a_2+b_2)\dots(a_n+b_n)]^{1/n}} \le \\
&\qquad \le \frac{1}{n}\left(\frac{a_1}{a_1+b_1} + \dots + \frac{a_n}{a_n+b_n}\right) + \frac{1}{n}\left(\frac{b_1}{a_1+b_1} + \dots + \frac{b_n}{a_n+b_n}\right) \\
&\qquad = \frac{1}{n}\left(\frac{a_1}{a_1+b_1} + \frac{b_1}{a_1+b_1} + \dots + \frac{a_n}{a_n+b_n} + \frac{b_n}{a_n+b_n}\right) \\
&\qquad = \frac{1}{n}\left(\frac{a_1+b_1}{a_1+b_1} + \dots + \frac{a_n+b_n}{a_n+b_n}\right) \\
&\qquad = \frac{1}{n}\cdot n = 1
\end{aligned}$$

which shows precisely what's required. □

10. (2004 Putnam) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} \le \frac{m! \cdot n!}{m^m n^n}$$

Hint: The fastest way to do this is far too clever for its own good and uses the binomial formula. However, there are many different methods!

Solution: The binomial formula states that

$$(m+n)^{m+n} = \sum_{i=0}^{m+n} \binom{m+n}{i} m^i n^{m+n-i}$$

Since m and n are positive integers, all the summands above are positive. Therefore, $(m+n)^{m+n}$ is larger than any one of the summands. In particular, letting $i = m$, we see that

$$\begin{aligned}
(m+n)^{m+n} &\ge \binom{m+n}{m} m^m n^{m+n-m} \\
&= \frac{(m+n)!}{m! \cdot n!} m^m n^n
\end{aligned}$$

Rearranging this inequality, we get

$$\frac{(m+n)!}{(m+n)^{m+n}} \le \frac{m! \cdot n!}{m^m n^n}$$

as required. □