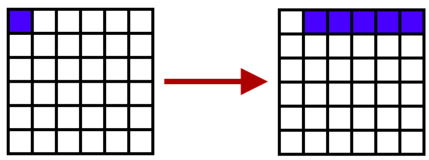
Solutions to Parity Problems:

1. In a 6×6 chart all but one corner blue square are painted white. You are allowed to repaint any column or any row in the chart (i.e., you can select any row or column and flip the color of all squares within that line). Is it possible to attain an entirely white chart by using only the permitted operations?

   Example operation:

   Solution: A little bit of experimenting will show that no matter how many moves you make, the number of blue squares will stay odd. (Try it yourself and see to check!)

   Why does that happen? To show that the number of squares stay odd, it suffices to show that it stays odd after any one operation. Let's try to prove that. An operation flips the color of every square in a given row or column. Let's say that before the operation, this row or column had x blue squares. In that case, it clearly had $6 - x$ white squares.

   Since the operation recolors every square, the number of blue squares in our row or column changes to $6 - x$. But note that

   $$(6 - x) - x = 6 - 2x = \text{an even number}$$

   which shows that x and $6 - x$ are of the same parity. This means our operation didn't change the parity of the number of squares in our chosen row or column. This means we didn't change the overall parity of the number of blue squares in the grid, and so we're done. □

   Note: I found it rather tedious to write 'row or column' every single time in the proof above! This is precisely why, when two possibilities are entirely symmetric, people tend to use the phrase "without loss of generality." In this proof, we'd say "without loss of generality, assume we're flipping all the squares in some row" and proceed just like before.

2. John and Pete have three pieces of paper. Each of the boys picks one piece, tears it up, and puts the smaller pieces back. John only tears a piece of paper into 3 smaller pieces while Pete only tears a piece of paper into 5 smaller pieces. After a few minutes can there be exactly 100 pieces of paper?

Sketch: Here, we would show that no matter what John and Pete do, the number of pieces of paper stays odd. We'd have to show that this happens no matter which one of them is doing the tearing – note that this works because both 3 and 5 are odd.

This shows that it's impossible that after a few minutes there are exactly 100 pieces of paper. It's also probably impossible that you could have some methodical little boys, but that's a separate issue... □

3. All natural numbers from 1 to 101 are written in a row. Can the signs "+" and "−" be placed between them so that the value of the resulting expression is 0?

   Sketch: Here, note that

$$1 + 2 + 3 + \cdots + 101 = \frac{101 \cdot 102}{2} = 101 \cdot 51 = \text{odd number}$$

   Furthermore, it is not difficult to show that changing a "+" to a "−" doesn't change the parity of the expression. Therefore, any number we can get by writing the numbers from 1 to 101 in a row and placing plus and minus signs between them must be odd. This means it can't possibly be equal to 0. □

4. Of 101 coins, 50 are counterfeit, and they differ from the genuine coins in weight by 1 gram. Peter has a scale in the form of a balance which shows the difference in weight between the objects placed on each pan. He chooses one coin, and wants to find out whether it is counterfeit. Can he do this in one weighing?

   Solution: Yes, he can. Here's what he can do: he can take the remaining 100 coins and split them however he likes into two groups of 50. He should then weigh the two groups against each other. If the difference between the weights of two groups is odd, then the coin is counterfeit; if the difference between the weights is even, then the coin is real. (And yes, the difference will definitely be an integer – more on that below.)

   To show how this works, we first establish some notation. Let the weight of a normal coin be w grams. Since we know that the weight of a counterfeit coin differs from w by 1 gram, for each coin, it's either $w - 1$ or $w + 1$.

   Say the two groups of 50 coins have weights $c_1, \ldots, c_{50}$ and $c_{51}, \ldots, c_{100}$, respectively. Our scale will show the difference between the weights of the groups: to be precise, we will know the value of

$$(c_1 + \cdots + c_{50}) - (c_{51} + ... + c_{100})$$

   Rearranging a bit, this is exactly the same as:

$$(c_1 - w) + (c_2 - w) + \cdots + (c_{50} - w) - (c_{51} - w) - \cdots - (c_{100} - w)$$

Note that this expression now uses only integers, since $c_i - w$ is an integer for all i. Since a number and its negative have precisely the same parity, the above has the same parity as

$$(c_1 - w) + (c_2 - w) + \cdots + (c_{100} - w)$$

But we know that $c_i - w$ is either 1 or -1 if coin i is counterfeit, and is 0 if coin i is real. Since both 1 and -1 are odd, while 0 is even, the parity of the above sum tells us precisely whether there's an even or an odd number of counterfeit coins in the 100 we're weighing. If there's an odd number of counterfeit coins being weighed, since the total number of counterfeit coins is even, the remaining 101st coin must be real. If there's an even number of counterfeit coins being weighed, we similarly conclude that the remaining 101st coin is real. This concludes the argument! $\square$

5. Suppose a, b and c are integers such that the equation $ax^2 + bx + c = 0$ has a rational solution. Prove that at least one of the integers a, b and c must be even.

   Solution: Let the rational solution of our equation be $\frac{p}{q}$, where p and q have no factors in common. Proceed by contradiction: assume that a, b and c are all odd. Plugging in $\frac{p}{q}$, we see that

$$a\frac{p^2}{q^2} + b\frac{p}{q} + c = 0$$

   Multiplying both sides by q^2, we get that

$$ap^2 + bpq + cq^2 = 0$$

   Now, since p and q have no factors in common, either they are both odd, or one of them is even. If p and q are both odd, then ap^2, bpq, and cq^2 are also all odd. A sum of three odd numbers can't be 0, so this is impossible.

   Therefore, we must have that one of p and q is even, and the other one is odd. Without loss of generality, assume that p is even (we aren't losing generality because our assumptions on a and c are identical, and therefore the above expression is symmetric in p and q.) Then, we have that ap^2 and bpq are even, while cq^2 is odd. Thus, we have a sum of two even numbers and an odd number, which also can't be 0.

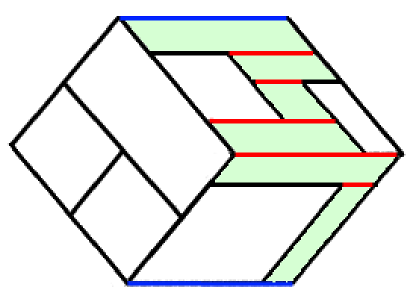
   We have now ruled out all possibilities, leading to a contradiction. Therefore, it's impossible that all of a, b and c are odd, and thus at least one of them must be even. $\square$

6. Can a convex nonagon (a polygon with 9 sides) be cut into parallelograms?

   Sketch: No, it can't. Assume that it is in fact possible, and proceed by contradiction. Now, we're able to pair up the sides of the nonagon in the following manner:

- Start at a side of the nonagon. Since we've split up the nonagon into parallelograms, this side must contain a side of a parallelogram. Jump to the opposite side of this parallelogram. (Here, and in the subsequent steps, you may have a choice of parallelograms – it doesn't matter how you make it.)
- We're now at a side of a parallelogram. If this is contained in another side of the nonagon, we're done. If not, then it must overlap with a side of another interior parallelogram which is across the line from the one we were at before. Jump to the opposite side of this parallelogram.
- This process will create a sequence of parallel sides: we will start at the side of the nonagon, and proceed to jump to sides of interior parallelograms. Since there are finitely many interior parallelograms, and we keep moving the same direction, we will eventually wind up at another side of the nonagon. Pair up this second side with the first one. We have now created a pair of parallel nonagon sides.

Here's an illustration of how this works in the case of a hexagon, which can indeed be split up into parallelograms. The blue lines are a pair 'matched up' hexagon sides, and the red lines illustrate the sequence of parallelogram sides connecting them; the parallelograms shaded in are the ones we're 'jumping across' when we construct our sequence of sides. Note that no matter which sequence of interior parallelogram sides you chose, the matching of the outer sides would be the same!



Note that this process by definition creates a pair of parallel nonagon sides. Since the nonagon is convex, we can't have any more than 2 sides parallel to one another. This means that we'll be able to pair up each side with precisely one other parallel side. However, this implies that the number of sides of a nonagon is even, which is clearly a contradiction!

By the way, note taht if the nonagon doesn't have to be convex, we'd be able to have more than 2 sides that are parallel to each other, and the above argument doesn't work. Indeed, there exist non-convex nonagons which can be cut into parallograms – try to think of example! □

7. Consider a football conference with 13 teams. Is it possible to schedule games so that each team plays exactly 9 games within the conference?

   Solution: It's impossible. Let's proceed by contradiction. Let g_i be the number of games played by team i. Note that in the sum $g_1+g_2+\cdots+g_13$, each game counted twice: if a game was between team i and team j, then it was counted both in g_i and g_j. Therefore, this sum must be even.

   However, we're assuming that $g_i = 9$ for each i in $\{1, 2, \ldots, 13\}$. This means that $g_1 + g_2 + \cdots + g_{13} = 9 \times 13$ which is odd. This is clearly impossible, so we get a contradiction. $\square$

8. (2002, A3) Let $n \geq 2$ be an integer and T_n be the number of nonempty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.
   Note: Ask me if you don't know what a set is!

   Solution: The idea here is to pair up the non-empty subsets of $\{1, 2, \ldots, n\}$ whose average is an integer.

   First of all, clearly the subsets $\{1\}, \{2\}, \{3\}, \ldots, \{n\}$ satisfy the property that the average of their elements is an integer. We're going to put these subsets aside. Since there are n of them, that leaves us precisely $T_n - n$ subsets to pair off.

   Here's how the pairing works. Let S is a subset whose average is an integer; let that average be called k. If S does not contain k, then pair it up with $S \cup \{k\}$; conversely, if S contains k, pair it up with $S/\{k\}$. Note that putting the average of a set into a set doesn't change the average, and neither does taking it away; as such, we're pairing up sets with the same average, and hence clearly are pairing up sets with integer averages. Furthermore, since we're avoiding the one-element sets, we won't pair up anything with the empty set.

   This pairing is easiest to see with an example. Let $n = 5$. Then, here's the pairing for all eligible sets:

$$\begin{aligned}
\{1,3\} &\leftrightarrow \{1,2,3\}\\
\{2,4\} &\leftrightarrow \{2,3,4\}\\
\{1,5\} &\leftrightarrow \{1,3,5\}\\
\{3,5\} &\leftrightarrow \{3,4,5\}\\
\{1,2,4,5\} &\leftrightarrow \{1,2,3,4,5\}
\end{aligned}$$

   As noted above, all we're doing to create pairs is to either take away the average from the set, or put it into the set. As yet another example, for $n = 7$, the set $\{2, 3, 7\}$ doesn't contain its average 4, and hence would be paired up with $\{2, 3, 4, 7\}$; on the other hand, the set $\{4, 5, 6\}$ contains its average 5, and hence would be paired up with $\{4, 6\}$.

Since we've managed to pair up all the subsets which don't contain one element, we've shown that $T_n - n$ is even, as required! $\square$

9. (2008, A2) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

   Note: Ask me if you don't know much about determinants: I can easily summarize all that's required here!

   Solution: Here, Barbara has the winning strategy. It's quite simple. Pair up column 1 with column 2. If Alan puts an x into the ith entry of column 1, Barbara counters this by putting an x into the ith entry of the column 2. Conversely, if Alan puts an x into the ith entry of column 2, Barbara will put an x into the ith entry of column 1. If Alan writes anything outside of columns 1 and 2, Barbara does whatever she wants, except that she's not allowed to write in columns 1 and 2 either.

   It should be clear that at the end of the game, this strategy leads to column 1 being precisely the same as column 2. But we know that matrices with two identical columns have determinant 0: therefore, Barbara wins! $\square$