The Pigeonhole Principle

1. Show that if we take $n+1$ numbers from the set $\{1, 2, \ldots, 2n\}$, then some pair of numbers will have no factors in common.

2. Show that if we take $n+1$ numbers from the set $\{1, 2, \ldots, 2n\}$, then there will be some pair in which one number is a multiple of the other one.

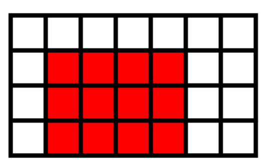
3. Given 5 points in the plane with integer coordinates, show that there exists a pair of points whose midpoint also has integer coordinates.

4. During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

5. Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most $\frac{1}{2}$.

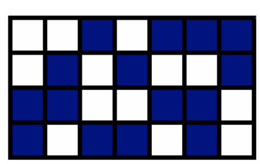
6. Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

7. A checkerboard has 4 rows and 7 columns. A *subboard* of a checkerboard is a board you can 'cut-out' of the checkerboard by only taking the squares which are between a specified pair of rows and a specified pair of columns. Here's an example of a subboard, with squares shaded in red:

   Now, suppose that each of the 28 squares is colored either blue or white. Show that there is a subboard all of whose corners are blue or all of whose corners are white.

   Here's an example of a coloring: see if you can find a subboard with four corners of the same color!

8. (2000 Putnam) Let a_j, b_j, c_j be integers for $1 \le j \le N$. Assume that for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $\frac{4N}{7}$ values of j, $1 \le j \le N$.

9. (2006 Putnam) Prove that for any set $X = \{x_1, x_2, \ldots, x_n\}$ of real numbers, there exists a non-empty subset S of x and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \le \frac{1}{n+1}$$

   Note: If you find the notation hard to read, come talk to me: it's actually not saying anything difficult at all!

10. (1993 Putnam) Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's