Pigeonhole Principle Solutions

1. Show that if we take $n+1$ numbers from the set $\{1, 2, \ldots, 2n\}$, then some pair of numbers will have no factors in common.

   Solution: Note that consecutive numbers (such as 3 and 4) don't have any factors in common. Therefore, it suffices to show that we'd have a pair of numbers that are consecutive.

   Let our pigeonholes be the following sets:

   $$\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}$$

   Here, our pigeons are the $n + 1$ numbers we're choosing from the set $\{1, 2, \ldots, 2n\}$. By the pigeonhole principle, two of our $n + 1$ numbers will be in the same pigeonhole – and since the above sets were chosen to contain pairs of consecutive numbers, this means that we'll have a pair of consecutive numbers. This means we'll have a pair of numbers with no factors in common. □

2. Show that if we take $n+1$ numbers from the set $\{1, 2, \ldots, 2n\}$, then there will be some pair in which one number is a multiple of the other one.

   Solution: Here, our pigeonholes are a little more complicated. To be precise, for each odd numbers $2m - 1$ in the set $\{1, 2, \ldots, 2n\}$, we make the set

   $$S_m = \{2m - 1, 2(2m - 1), 4(2m - 1), \ldots, 2^k(2m - 1), \ldots\}$$

   That is, S_m contains the odd number $2m - 1$ and any number that can be obtained by multiplying $2m - 1$ by a power of 2. Then, our pigeonholes are defined to be the sets $S_1, S_2, \ldots, S_n$.

   The above is a lot of notation. The easiest thing to do is to make up an example to illustrate what's going on. For example, for $n = 4$, the pigeonholes are:

   $$S_1 = \{1, 2, 4, 8\}$$
   $$S_2 = \{3, 6\}$$
   $$S_3 = \{5\}$$
   $$S_4 = \{7\}$$

For a more complicated example, if $n = 7$, the pigeonholes are:

$$S_1 = \{1, 2, 4, 8\}$$
$$S_2 = \{3, 6, 12\}$$
$$S_3 = \{5, 10\}$$
$$S_4 = \{7, 14\}$$
$$S_5 = \{9\}$$
$$S_6 = \{11\}$$
$$S_7 = \{13\}$$

Now, note that we have n pigeonholes and $n + 1$ pigeons – that is, we're picking $n + 1$ numbers. This means that a pair of the numbers will be in the same set S_m. But the sets S_m were carefully chosen so that out of any pair of numbers in the same set, one number divides the other. Thus, the presence of a pair of numbers in same S_m ensures that we will have two numbers one of which is a multiple of the other, as required. □

3. Given 5 points in the plane with integer coordinates, show that there exists a pair of points whose midpoint also has integer coordinates.

   Solution: If (a, b) and (c, d) are two points in the plane, then the midpoint is the point $(\frac{a+c}{2}, \frac{b+d}{2})$. This point is an integer precisely if $a + c$ and $b + d$ are both even. After a little bit of thought, it's clear that this happens if both the following statements hold: a and c have the same parity, and b and d have the same parity.

   Let the pigeonholes be defined by the parity of the two numbers. That is, the pigeonholes are:

   $$S_{OO} = \{(x, y) \mid x \text{ odd}, y \text{ odd}\}$$
   $$S_{OE} = \{(x, y) \mid x \text{ odd}, y \text{ even}\}$$
   $$S_{EO} = \{(x, y) \mid x \text{ even}, y \text{ odd}\}$$
   $$S_{EE} = \{(x, y) \mid x \text{ even}, y \text{ even}\}$$

   So, for example, $(1, 1)$ is in S_{OO}, and $(2, 1)$ is in S_{EO}.

   Since we're picking 5 points, we'll have two points in the same pigeonhole. Therefore, we will have two points (a, b) and (c, d) such that a and c as well as b and d are the same parity. From our arguments above, this will mean that the midpoint between them will be an integer, as required. □

4. During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: For this question, it helps to set up some notation. Let the number of games played on day i be g_i. Then, the total number of games played starting on day m, and ending on day n is

$$g_m + g_{m+1} + \cdots + g_n$$

Now, define s_i to be the total number of games played during the first i days. That is,

$$s_1 = g_1$$
$$s_2 = g_1 + g_2$$
$$\vdots$$
$$s_{30} = g_1 + g_2 + \cdots + g_{30}$$

Now, note that

$$g_m + g_{m+1} + \cdots + g_n = (g_1 + \cdots + g_n) - (g_1 + \cdots g_{m-1})$$
$$= s_n - s_{m-1}$$

Therefore, what we need to show that is that $s_n - s_{m-1}$ takes on the value 14 for some m and n between 1 and 30. What we're given is that the total number of games played on all thirty days is no more than 45, which tells us that each s_i is at most 45. Furthermore, we're told that the team plays at least a game a day, which means that $s_1, s_2, \cdots, s_{30}$ are distinct integers between 1 and 45. The s_i are going to be the pigeons in this question.

Now, let our pigeonholes be the following sets:

$$\{1, 15\}, \{2, 16\}, \ldots, \{14, 28\}$$
$$\{29, 43\}, \{30, 44\}, \{31, 45\}$$
$$\{32\}, \{33\}, \ldots, \{42\}$$

We're picking 30 numbers $s_1, s_2, \ldots, s_{30}$ out of the set $\{1, 2, \ldots, 45\}$. Counting, we see that that we have exactly $14+3+11 = 28$ pigeonholes. Therefore, some two of the s_i are going to be in the same pigeonhole. By our definition, a pair of numbers in a pigeonhole differ by exactly 14 – this means we'll have an s_i and s_j such that $s_j - s_i = 14$. But as noted above, this means that the total number of games played on days $i+1, i+2, \ldots, j$ is exactly 14, as required. $\square$

5. Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most $\frac{1}{2}$.

   Solution: This question is best solved with a picture. Split up the equilateral triangle into 4 little equilateral triangles of side $\frac{1}{2}$ as shown here:

3
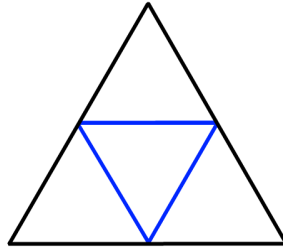
It should be clear that any two points within one of the smaller 4 triangles are within distance $\frac{1}{2}$. Since we have 5 points in the bigger triangle, some two of them will necessarily be in the same small triangle – hence, that pair of points will have distance at most $\frac{1}{2}$, and we're done. $\qquad \square$

6. Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

Solution: Let the ten numbers be $a_1, a_2, \ldots, a_{10}$. For any subset T of $\{1, 2, \ldots, 10\}$, let

$$S_T = \sum_{i \in T} a_i$$

For example,

$$S_{\{1,2,4\}} = a_1 + a_2 + a_4$$
$$S_{\{2,5,8\}} = a_2 + a_5 + a_8$$

Since the a_i are two-digit numbers, and each sum contains at most 10 terms, we see that S_T is between 10 and 990 for each subset T. Now, the number of subsets of $\{1, 2, \ldots, 10\}$ is precisely $2^{10} = 1024$, and therefore the number of non-empty subsets is 1023. Since there are fewer than 1023 choices for possible values of S_T, this means that there exist subsets T and R of $\{1, 2, \ldots, 10\}$ such that

$$S_T = S_R$$

If R and T are disjoint, then we're done. However, if they overlap, we can just subtract the overlapping items from both sides. For example, if we have that $S_{\{1,3,4,5\}} = S_{\{3,5,7,8\}}$ then

$$a_1 + a_3 + a_4 + a_5 = a_3 + a_5 + a_7 + a_8$$
$$\Rightarrow a_1 + a_4 = a_7 + a_8$$

Thus, this will allow us to find two disjoint subsets of the 10 numbers with the same sum. $\qquad \square$

7. A checkerboard has 4 rows and 7 columns. A *subboard* of a checkerboard is a board you can 'cut-out' of the checkerboard by only taking the squares which are between a specified pair of rows and a specified pair of columns. Here's an example of a subboard, with squares shaded in red:
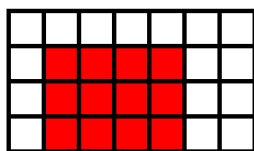
Now, suppose that each of the 28 squares is colored either blue or white. Show that there is a subboard all of whose corners are blue or all of whose corners are white.

Here's an example of a coloring: see if you can find a subboard with four corners of the same color!
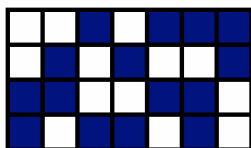
Solution: Let us first discuss how to approach this. Note that we get a subboard whose corners are the same color if two columns 'agree' on two squares: for example, if column 2 and column 5 both have blue squares in the 1nd and 4th rows, then we'd get the following (the squares for which we don't know the colors are colored in light blue):
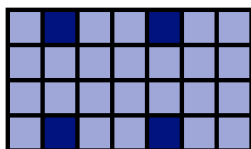
As another example, in the coloring given in the question, columns 1 and 6 both have blue squares in the 3rd and 4th row, which similarly leads to a desired subboard.

Let us proceed by contradiction: assume that there exists a coloring such that no subboard has all four corners of the same color. We need to consider a couple of cases. Let's consider the first column of the checkerboard. Either the first column has 2 of each color, or we have at least 3 squares of

one color (either blue or white.) Since there's no difference between blue and white in the context of the problem, let's start with the case where we have at least 3 blue squares in the first column.

Case 1: 3 blue squares in the first column. Since permuting the rows doesn't change anything, let's assume that the first column has blue squares in rows $1, 2$ and 3. As noted above, if any of the remaining columns have blue squares in 2 of those rows, we'd have a subboard with matching corners; for example, if column 5 has 2 blue squares in the first 3 rows, here's what can happen:
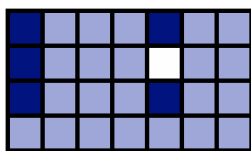
Therefore, we need to assume that each of the remaining 6 columns each have at least 2 white squares in the first 3 rows; here's an example of such a coloring: (note that for now, we're ignoring the fourth row, hence the light blue squares)
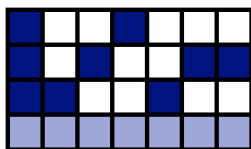

As should be clear, we're now having an issue with having a subboard all of whose corners are white. Indeed, there are only 4 ways to color in the first 3 rows of a column by using at most 1 blue square; here they are:
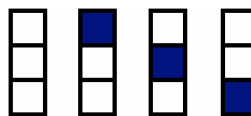
By the pigeonhole principle, since there are only 4 potential colorings, and 6 columns to color in, some two columns will agree on the first 3 rows. This means that we'll have a subboard all of whose corners are white, as required.

Case 2: 2 blue squares in the first column. First, note that the argument above would hold even if we assumed that another column had 3 blue squares. (This is because permuting rows and permuting columns doesn't change anything.) Therefore, for this case we may assume that

6

every single column has exactly 2 blue squares and 2 white squares. There are exactly 6 ways to color in a column in this way; here they are:
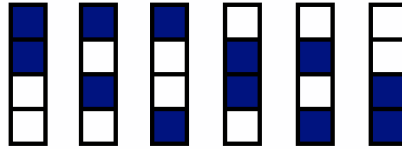
Since we have 7 columns, but only 6 ways of coloring them, two columns will be colored in the exact same fashion. This means that we will again have a subboard all of whose corners are the same color, and we're done.
□

The Remaining Problems: I didn't see too many people try the last three problems; besides, they are Putnam problems and it's easy enough to find their solutions online. So I'm not going to post solutions to them yet – let me know if you've tried them and really want to see how they work!