Homework 6

1. In this question, we will complete the proof that for an irreducible and aperiodic Markov chain with stationary distribution π and transition matrix P, there exist constants $C > 0$ and α between 0 and 1 such that

$$\max_{x \in \Omega} \|P^t(x, \cdot) - \pi\| \leq C\alpha^t \tag{1}$$

   Recall that for our choice of r and Π, we have shown that for all integers k and j

$$P^{rk+j} - \Pi = \theta^k(Q^k P^j - \Pi)$$

   (a) [5 pts] Use the equation above to show that

$$\max_{x \in \Omega} \|P^{rk+j}(x, \cdot) - \pi\|_{TV} \leq \theta^k$$

   (Feel free to use the fact that $Q^k P^j$ and Π are both stochastic matrices.)

   (b) [5 pts] Now, let t be any positive integer. Use part (a) to show that

$$\max_{x \in \Omega} \|P^t(x, \cdot) - \pi\|_{TV} \leq \theta^{\lfloor t/r \rfloor}$$

   Notation: $\lfloor \cdot \rfloor$ is the "floor" or "greatest integer less than" function. As you may have figured out from that name, $\lfloor x \rfloor$ is the greatest integer less than x. For example, $\lfloor 1.5 \rfloor = 1$, $\lfloor 2.99 \rfloor = 2$, $\lfloor -1.3 \rfloor = -2$.

   (c) [5 pts] Define $\alpha = \theta^{1/r}$ and $C = 1/\theta$. Use part (b) to show that Equation (1) holds with this choice of α and C, and that α is between 0 and 1 and $C > 0$.

2. In this question, we will be doing a bit of coding and using a number of new Matlab functions. You can find references for all of these at http://www.mathworks.com/help/ by typing the search term into the box that says 'Search R2012b Documentation' and narrowing down to results restricted to Matlab on the lefthand side after you enter the search term. If you need more help, send m an e-mail and I'll be happy to explain more!

   Print out your Matlab code so that I can grade it for the questions below; make sure that Matlab outputs the appropriate answers so that I can see them.

   Tell them to look at the video about the coding environment?

   (a) [5 pts] Use a 'for loop' to construct the transition matrix P for the lazy simple random walk on the 10-cycle.

   Hint: Matlab initializes all matrix entries as 0, so you only need to define the non-zero entries – therefore, you want to set $P(i, i-1), P(i, i)$ and $P(i, i+1)$ to their values for all appropriate i. You may also want to do the first and last row separately!

(b) [5 pts] Use a 'for loop' to define a distribution π on $\{1, \ldots, 10\}$ that is 1/10 on each state. Verify that π is stationary for P.

(c) [2 pts] Now, use the Matlab 'ones' function to define a matrix Π of size 10×10 whose each entry is 1/10.

Note: We could have done the same thing to define the π in part (b) – Matlab is optimized so that this is much faster than using a 'for loop'.

(d) [5 pts] Use the Matlab 'abs' function to compute a matrix A whose (i, j) entry is $|P^3(i, j) - \Pi(i, j)|$. Now, use the 'sum' and 'max' functions to calculate

$$\max_{x \in \Omega} \|P^3(x, \cdot) - \pi\|_{TV}$$

(e) [5 pts] Use a 'for loop' to calculate a vector V of length 20 whose ith entry is $\max_{x \in \Omega} \|P^i(x, \cdot) - \pi\|_{TV}$. (If you figured out how to do part (d), this should be a very short piece of code!)

(f) [3 pts] Use part (e) to calculate $t_{\text{mix}}(1/4), t_{\text{mix}}(1/8)$ and $t_{\text{mix}}(1/10)$.