Homework 8

1. (a) [5 pts] Let μ and ν be two distributions on Ω, and let (X, Y) be a coupling of μ and ν. Show that for all $A \subset \Omega$,

$$\mu(A) - \nu(A) \leq \mathbb{P}\{X \neq Y\}$$

   (b) [3 pts] Use part (a) to show that $\|\mu - \nu\|_{TV} \leq \mathbb{P}\{X \neq Y\}$.

2. Consider the following coupling of the random transposition walk: we pick a label l and a position p, and transpose the card labelled l with the card in position p in both chains. (For example, if we're at $(1234, 4321)$ then if we transpose position 3 with label 1 the coupling goes to $(3214, 4312)$.) Let the coupling be as usual denoted by (X_t, Y_t).

   (a) [5 pts] Let D_t be the number of unmatched cards for the pair (X_t, Y_t): that is, the number of cards whose positions are different in the two chains. (For example, at $(3214, 4312)$ the number of unmatched cards is 3 since the positions of 2, 3 and 4 are different in both chains.) Show that $D_{t+1} \leq D_t$.

   (b) [5 pts] Show that $\mathbb{P}\{D_{t+1} \leq i - 1 \mid D_t = i\} = i^2/n^2$.

   (c) [5 pts] Let τ_i be the first time $D_t \leq i$. Show that if we condition on $\tau_{i-1} \neq \tau_i$, $\tau_{i-1} - \tau_i$ is distributed as a geometric random variable with parameter i^2/n^2.

   (d) [5 pts] Clearly, τ_0 is the first time $D_t = 0$ and hence $X_t = Y_t$. Use part (c) to show that

$$\mathbb{E}(\tau_0) \leq \sum_{i=1}^{\infty} \frac{n^2}{i^2}$$

   (e) [5 pts] Use part (d) to show that

$$\mathbb{E}[T^{x,y}_{couple}] \leq 2n^2$$

   (f) [5 pts] Use part (e) and Markov's inequality to show that t_{mix} is at most of order n^2.

3. In this question, we will be simulating the above coupling for the random transposition walk. We will be needing to use a pseudo-random number generator in Matlab (it's pseudo-random since it's very hard to come up with truly random numbers.) The relevant functions are rng and randi – you may want to check the Matlab documentation for them.

   For our purposes, we will want to use the Mersenne Twister pseudo-random number generator. Furthermore, since pseudo-random numbers need a 'seed' (Google this if you're curious!) we will, as is standard, use the current time as the seed. This will ensure that the pseudo-random number generator gives a different number each time. To accomplish these, we will always start code that uses pseudo-random numbers with rng('shuffle', 'twister').

For the following questions, you will only need the fact that randi(n) generates a pseudo-random integer between 1 and n.

(a) [5 pts] Writing a permutation as a vector, write the code that swaps a random pair of cards in a permutation of length 10.

(b) [5 pts] Now, write code that performs the random transposition walk for 200 steps.

(c) [5 pts] Write code that simulates the coupling in question 2 for 30 steps.

(d) [5 pts] Run the code in part (c) 100 times, and keep track of the percentage of the time that the coupling meets.