Finite dimensional approximation and Conley Index

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October 4, 2019

Abstract

In this seminar I will talk about the Conley Index section in Manolescu's paper "Lectures on the Triangulation Conjecture". I don't claim any originality of this work, all errors are mine.

1 Compactness of the Moduli space of solutions

In the last talk, we proved that we can write

$$SW = l + c\colon V \to V \tag{1}$$

where V is our Coulomb slice, and the functions l, c are given as follows:

$$\begin{aligned} l(a,\phi) &= (*da, \not{\partial}\phi) \\ c(a,\phi) &= \pi_V \circ (-2\rho^{-1}(\phi \otimes \phi^*)_\circ, \rho(a)\phi) \end{aligned}$$

Let $V_{(k)}$ be the $W^{2,k}$-completion of V for a fixed number $k \gg 0$. We will take $k > 5$. Then, the map $l\colon V_{(k)} \to V_{(k-1)}$ is a linear, self-adjoint, Fredholm operator, and $c\colon V_{(k)} \to V_{(k-1)}$ is a compact map.
The following is the standard compactness theorem for Seiberg-Witten equations, adapted to Coulomb gauge.

Theorem 1.1. *Fix $k > 5$. There exists some $R > 0$ such that all the critical points and flow lines between critical points of SW are contained inside the ball $B(R) \subset V_{(k)}$.*

2 Finite dimensional approximation

Seiberg-Witten Floer homology is meant to be Morse homology for the functional SW on V. However, instead of finding a generic perturbation to achieve transversality, it is more convenient to do finite dimensional approximation. In the finite dimensional case, we can simply use singular homology instead of Morse homology.
Our finite dimensional approximation is inspired by Furuta's 4-dimensional case. In our setting V is an infinite dimensional space, and as a finite dimensional approximation of V we consider

$$V_\lambda^\mu = \bigoplus (\text{eigenspaces of } l \text{ with eigenvalues in } (\lambda,\mu)), \quad \lambda \ll 0 \ll \mu$$

We replace $SW = l + c\colon V \to V$ by

$$l + p_\lambda^\mu c\colon V_\lambda^\mu \to V_\lambda^\mu$$

where p_λ^μ is the L^2-projection onto V_λ^μ. Then,

$$SW_\lambda^\mu = l + p_\lambda^\mu c$$

is a vector field on V_λ^μ.
The following is a compactness theorem in the finite dimensional approximations.

Theorem 2.1. *There exists $R > 0$ such that for all $\mu \gg 0 \gg \lambda$ all critical points of SW_λ^μ in the ball $B(2R)$, and all flow lines between them that lie in $B(2R)$, actually stay in the smaller ball $B(R)$.*

Seiberg-Witten trajectories $x(t)$ can be interpreted in a standard way as solutions of the four-dimensional monopole equations on the cylinder $\mathbb{R} \times Y$.

Proof. The idea of the proof is to use that in $B(2R)$, $l + p_\lambda^\mu c$ converges uniformly to $l + c$. We then apply Theorem 1.1 □

3 The Conley Index

We start with a very brief introduction and examples about Morse Theory. Let M be a closed Riemannian manifold. Given a Morse-Smale function $f : M \to \mathbb{R}$, there is an associated Morse complex $C_*(M, f)$. The generators are the critical points of f and the differential is given by

$$\partial x = \sum_y n_{xy} y$$

where n_{xy} is the signed count of index 1 gradient flow lines between x and y. The Morse homology $H_*(M, f)$ is isomorphic to the usual singular homology of M.
Let us investigate what happens if we drop the compactness assumption. In general, it may no longer be the case that $\partial^2 = 0$:

Example 3.1. Let us consider the Heart-shaped sphere depicted in 1.

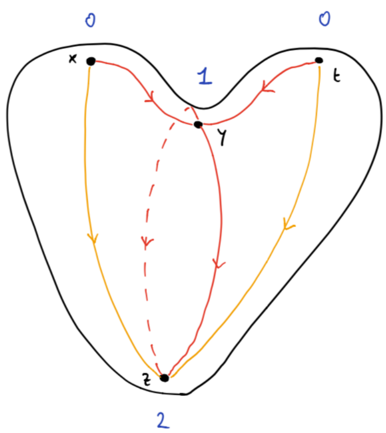

Figure 1: A Sphere

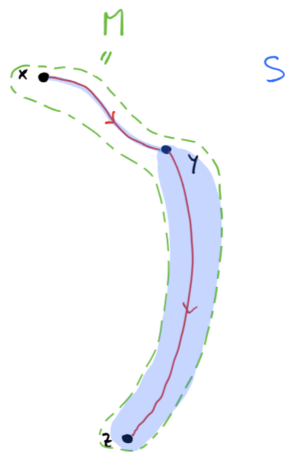

Figure 2: Flow lines

Suppose we have a Morse-Smale function f on the surface, a gradient flow line from a local maximum to a saddle point y, and another flow line from y to a local minimum z. Let M be a small open neighbourhood of the union of these two flow lines, as depicted in Figure 2. Then the restriction of f to M does not yield a Morse chain complex: we have $\partial^2 x = \pm \partial y = \pm z$.

In order to obtain a Morse complex on a non-compact manifold M, we need to impose an additional condition. Let $f : M \to \mathbb{R}$ be Morse-Smale. Some gradient flow lines of f connect critical points, while others may escape to the ends of the manifold, in positive and/or negative time (and either in finite or infinite time). Let us denote by $\mathcal{S} \subseteq M$ the subset of all points that lie on the flow lines connecting critical points (In particular, $\mathcal{S}$ includes all the critical points). In Example 3.1, the set $\mathcal{S}$ is not compact. This is related to the non-vanishing of ∂^2: the moduli space of flow line through y only gives a partial compactification of M (by a single point).
Therefore let us assume that $\mathcal{S}$ is compact. Then the same proof as in the case when M is compact

shows that the differential ∂ satisfies $\partial^2 = 0$. We obtain a Morse complex $C_*(M, f)$. The next question is, what does the Morse homology $H_*(M, f)$ compute in this case? As the reader can check in simple examples, it does not give the singular homology of either M or $\mathcal{S}$. The answer turns out to be the homology of the Conley index of $\mathcal{S}$, which we now proceed to define.

Definition 3.2. For a subset $A \subseteq M$ we define

$$\text{Inv}A = \{x \in A \mid \phi_t(x) \in A \ \forall t \in \mathbb{R}\}$$

Definition 3.3. A compact subset $S \subset M$ is called an isolated invariant if there exists A, a compact neighbourhood of S such that

$$S = \text{Inv}A \subseteq \text{Int}A$$

Definition 3.4. For an isolated invariant set S, the Conley index is defined as the pointed homotopy type of $I(S) = (N/L, [L])$ where $L \subseteq N \subseteq M$, with both L and N compact and such that they satisfy

1. $\text{Inv}(N \setminus L) = S \subseteq \text{Int}(N \setminus L)$
2. L is an exit set for N; that is, for all $x \in N$, if there exists $t > 0$ such that $\phi_t(x)$ is not in N, then there exists $0 \leq \tau < t$ with $\phi_\tau(x) \in L$.
3. L is positively invariant in N; that is, if $x \in L$ and $t > 0$ are such that $\phi_s(x) \in N$ for all $0 \leq s \leq t$, then $\phi_s(x)$ is in L for $0 \leq s \leq t$.

The pair (N, L) is called an index pair. It was proved by Conley that any isolated invariant set $\mathcal{S}$ admits an index pair.

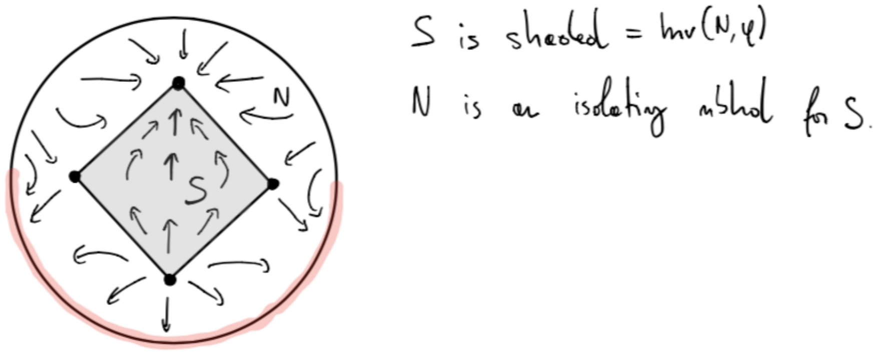

Figure 3: A example of a Conley Index

Example 3.5. Suppose ϕ is the downward gradient flow of a Morse function, and $\mathcal{S} = \{x\}$ consists of a single critical point of Morse index k. We can find an isolating neighborhood N for $\{x\}$ of the form $D^k \times D^{n-k}$, with $L = \partial D^k \times D^{n-k}$ being the exit set (See figure 4). We deduce that the Conley Index of $\{x\}$ is the homotopy type of S^k, so k can be recovered from $I(\phi, \mathcal{S})$. Thus we can view the Conley index as a generalization of the usual Morse index.

Theorem 3.6. *The Conley Index $I(\phi, \mathcal{S})$ is an invariant of the triple $(M, \phi, \mathcal{S})$. The Conley index is invariant under continuation: if we have a smooth family of flows $\phi^\lambda = \{\phi_t^\lambda\}$, $\lambda \in [0, 1]$ and N is an isolating neighbourhood in every ϕ^λ, then the Conley index for $\mathcal{S}_\lambda = Inv(N, \phi_\lambda)$ in the flow ϕ^λ is independent of λ.*

In practice, it is helpful to know that we can find index pairs with certain nice properties. For any isolated invariant set $\mathcal{S}$, we can choose an index pair (N, L) such that N and L are fnite CW complexes. In fact, more is true: we can arrange so that N is an n-dimensional manifold with boundary and $L \subset \partial N$ is an $(n-1)$-dimensional manifold with boundary.

Figure 4: The index pair (N, L) for a critical point of index 1 for a flow on a surface

3.1 The equivariant Conley index

Floer and Pruszko refined Conley index theory to the equivariant setting. Precisely, let G be a compact Lie group acting smoothly on a manifold M, preserving a flow ϕ and an isolated invariant set $\mathcal{S}$. Then, there exists a G-invariant index pair (N, L) for $\mathcal{S}$, and the Conley index

$$I_G(\phi, \mathcal{S}) := (N/L, [L])$$

is well-defined up to canonical G-equivariant homotopy equivalence.

4 Seiberg-Witten Floer Homology

Remember that $\mathrm{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{C} \cup j\mathbb{C} = \mathbb{H}$ and j acts on V as an additional symmetry as follows:

$$j : (a, \phi) \mapsto (-a, \phi j)$$

$$\Gamma(S) = \{U \to \mathbb{C}^2\}$$
$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -\bar{v}_2 \\ -\bar{v}_1 \end{pmatrix}$$

In order to apply the Conley Index machinery in our case, we take $A = B(2R)$, so that $\mathcal{S} = \mathrm{Inv}A$ (the union of the critical points and flow lines inside the ball). Then, N can be taken to be a manifold with boundary and $L \subset \partial N$ a codimension 0 submanifold of the boundary (so that L has its own boundary). It can be shown that, if the flow lines satisfy the Morse-Smale condition, then the Morse homology is isomorphic to the reduced singular homology of $I(S)$.

Definition 4.1. We define the S^1-equivariant Seiberg-Witten Floer homology of a homology sphere Y to be

$$SWFH_*^{S^1}(Y) = \widetilde{H}^{S^1}_{*+\text{ shift}}\left(I^\mu_\lambda\right) \quad \text{for } \mu \gg 0 \gg \lambda$$

Similarly, for the Pin(2) equivariant SWF homology (the SW equations are Pin(2)-equivariant, thus we can give the definition of Pin(2)-equivariant Seiberg-Witten Floer homology) we have:

$$SWFH_*^{\mathrm{Pin}(2)}(Y; \mathbb{F}) = \widetilde{H}^{\mathrm{Pin}(2)}_{*+\text{ shift}}(I^\mu_\lambda; \mathbb{F}) \quad \text{for } \mu \gg 0 \gg \lambda$$

Here, I^μ_λ is Conley index for $\mathcal{S}^\mu \subset V^\mu_\lambda$, and the grading is shifted by some amount that depends on λ, μ, and the metric. Note that everything in the construction is S^1-equivariant (resp. Pin(2)), with respect to the S^1-action by constant gauge transformation (resp. Pin(2) with the appropriate action). This allow us to apply Borel homology in the formula above.

4.0.1 Invariance

Let us prove that the Floer homologies we have just defined, $SWFH_*^{S^1}(Y;\mathbb{F})$, $SWFH_*^{\mathrm{Pin}(2)}(Y;\mathbb{F})$, are invariants of Y. In the process, we will also identify the grading shift in their definitions.
Remember we have a map $SW\colon V \to V$, where $V = \ker d^* \oplus \Gamma(S)$ and V_λ^μ is a finite dimensional approximation of V. Then the SW-equations can be decomposed as $SW = l + c$, with approximations

$$SW_\lambda^\mu = l + p_\lambda^\mu c\colon V_\lambda^\mu \to V_\lambda^\mu$$

The flow equation is

$$\dot{x} = -SW_\lambda^\mu(x(t))$$

Let us investigate how the Conley index I_λ^μ changes under varying μ and λ. If we change $\mu \rightsquigarrow \mu' > \mu \gg 0$, we have a decomposition

$$V_\lambda^{\mu'} = V_\lambda^\mu \oplus V_\mu^{\mu'}$$

which induces

$$l + p_\lambda^{\mu'} c \rightsquigarrow l + p_\lambda^\mu c \oplus l + p_\mu^{\mu'} c$$

The Conley index is invariant under deformations i.e., if we have a family of flows $\phi(s)$ where $s \in [0,1]$ such that

$$S(s) = \mathrm{Inv}(B(R) \text{ in } \phi(s)) \subset \mathrm{Int}B(R), \quad \text{where } s \in [0,1]$$

then $I(S(0)) \simeq I(S(1))$.

Claim 4.2. *The flow of $l + p_\mu^{\mu'} c$ is isotopic to the flow of l on $V_\mu^{\mu'}$*

Proof. Since we are working on $V_\mu^{\mu'}$ l can be easily isotope to the identity by rescaling the eigenvalues. Now we use the fact that a compact perturbation of the identity is invertible to prove that $l + tp_\mu^{\mu'} c$ for $t \in [0,1]$ is an isotopy from $l + p_\mu^{\mu'} c$ to l. Since the Conley index is invariant by perturbations we can modify everything up to isotopy □

In our case, we let $\phi(0)$ be the flow of $l + p_\lambda^{\mu'} c$ and deform it into $\phi(1)$, the direct sum of the flow of $l + p_\lambda^\mu$ and the linear flow l on $V_\mu^{\mu'}$. We get

$$I_\lambda^{\mu'} = I(S(0)) = I(S(1)) = I_\lambda^\mu \wedge I_\mu^{\mu'}(l)$$

Here, $I_\mu^{\mu'}(l)$ is the Conley Index for the linear flow $\dot{x} = -l(x)$ on $V_\mu^{\mu'}$. Since the restriction of l to that subspace has only positive eigenvalues, we see that

$$I_\mu^{\mu'}(l) = S^{\text{Morse index}} = S^0$$

We obtain

$$I_\lambda^{\mu'} = I_\lambda^\mu \;\text{ when } \mu, \mu' \gg 0$$

On the other hand, by a similar argument, when we vary the cut-off λ for negative eigenvalues, the Conley index changes by the formula

$$I_{\lambda'}^\mu = I_\lambda^\mu \wedge \underbrace{\left(V_{\lambda'}^\lambda\right)}_{\text{sphere}}$$

We conclude that:

$$\widetilde{H}^{S^1}_{*+\dim V_\lambda^0}(I_\lambda^\mu)$$

is independent of λ and μ, provided $\mu \gg 0 \gg \lambda$. The same is true for the Pin(2)-equivariant homology. This suggests including a degree shift $\dim V_\lambda^0$ in the definitions.