# Additional questions for Homework 3

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Recall that the Mandelbrot set $M$ is defined as the set of those $c \in \mathbb{C}$ with the property that the sequence $(z_n(c))$ is bounded. Here $z_n(c)$ is defined recursively by $z_0(c) = 0$ and $z_{n+1}(c) = z_n(c)^2 + c$.

1. In this question we will show that $M$ is contained in the closed disc $\overline{D(0;2)} = \{z \in \mathbb{C} : |z| \leq 2\}$.

   (a) Let $(a_n)$ be a sequence of complex numbers. Show that if $a_n \to \infty$ then $a_n$ is not bounded.

   (b) Suppose $|c| > 2$. We aim to show that $z_n(c) \to \infty$. To this end, prove that for each $n \geq 2$,

   $$|z_n(c)| \geq r_n |c|,$$

   where the sequence $r_n$ is defined recursively starting at $n = 2$ by $r_2 = |c| - 1$ and $r_{n+1} = 2r_n^2 - 1$. [*Hint*: use induction and the triangle inequality in the form $|a - b| \geq |a| - |b|$.]

   (c) Write $|c| = 2 + \epsilon$. Prove that $r_n \geq 1 + 4^{n-2}\epsilon$ for $n \geq 2$.

   (d) Prove that $r_n \to \infty$ (i.e., for all $R$ there's some $N$ such that $|r_n| > R$ whenever $n > N$).

   (e) Use the last part to help you prove that $z_n(c) \to \infty$. Deduce that $c \notin M$.

2. In this question we show that, for a given $c \in \mathbb{C}$, if there's some $N$ for which $|z_N(c)| > 2$, then $z_n(c) \to \infty$ as $n \to \infty$ (so $c \notin M$). This is very useful when drawing computer pictures of the Mandelbrot set.

   By the previous question, we know that $z_n(c) \to \infty$ when $|c| > 2$, so we assume $|c| \leq 2$.

   (a) Prove that for each integer $k \geq 0$ we have $|z_{N+k}(c)| \geq 2 + 4^k\epsilon$.

   (b) Prove that $z_n(c) \to \infty$.