M361 Theory of functions of a complex variable: Algebra and geometry of the complex numbers

T. Perutz

U.T. Austin, Fall 2012

1 Lecture 1. The complex numbers viewed as $\mathbb{R}^2$ with a vector product

1.1 Vector products

The system of complex numbers is just the real plane $\mathbb{R}^2$ with a special vector product. To set it in context, let's look at vector products more generally.

Let n be a positive integer, and let $\mathbb{R}^n$ be the set (or *vector space*) of real n-vectors. Members of $\mathbb{R}^n$ are lists $(x_1, \ldots, x_n)$ of n real numbers. We have an addition rule $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$, and a rule for scaling a vector by a real number, $s \cdot (x_1, \ldots, x_n) = (sx_1, \ldots, sx_n)$.

Definition 1.1. A **vector product** on $\mathbb{R}^n$ is a binary operation $\star$ on $\mathbb{R}^n$. So for each pair of inputs $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, we have $\mathbf{x} \star \mathbf{y} \in \mathbb{R}^n$. We require that $\star$ should be distributive (on both sides):

$$\mathbf{x} \star (a\mathbf{y} + b\mathbf{z}) = a\mathbf{x} \star \mathbf{y} + b\mathbf{x} \star \mathbf{z},$$

$$(a\mathbf{x} + b\mathbf{y}) \star \mathbf{z} = a\mathbf{x} \star \mathbf{z} + b\mathbf{y} \star \mathbf{z}.$$

Vector products are very easy to construct. Take the standard basis vectors for $\mathbb{R}^n$:

$$\mathbf{e}_j = (0, \ldots, 0, \underset{j}{1}, 0, \ldots, 0), \quad j = 1, \ldots, n.$$

For each pair (j, k), decide what you want $\mathbf{e}_j \star \mathbf{e}_k$ to be (it can be any vector in $\mathbb{R}^n$). Then define

$$(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \star (y_1\mathbf{e}_1 + \cdots + y_n\mathbf{e}_n) = \sum_{j,k} x_j y_k (\mathbf{e}_j \star \mathbf{e}_k).$$

Example 1.2. In $\mathbb{R}^2$, let's define $\mathbf{e}_1 \star \mathbf{e}_1 = \mathbf{e}_1$, $\mathbf{e}_1 \star \mathbf{e}_2 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_2 \star \mathbf{e}_1 = 3\mathbf{e}_2$, $\mathbf{e}_2 \star \mathbf{e}_2 = \mathbf{0}$. Then

$$\begin{aligned}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \star (y_1\mathbf{e}_1 + y_2\mathbf{e}_2) &= x_1y_1\mathbf{e}_1 + x_1y_2(\mathbf{e}_1 + \mathbf{e}_2) + 3x_2y_1\mathbf{e}_2 \\ &= (x_1y_1 + x_1y_2)\mathbf{e}_1 + (x_1y_2 + 3x_2y_1)\mathbf{e}_2.\end{aligned}$$

With such a profusion of vector products, we need a reason to single one out as special. There are several possible criteria, and we'll only mention a couple.

The **norm** (or **length**) of a vector $\mathbf{x} \in \mathbb{R}^n$ is

$$|\mathbf{x}| := \sqrt{x_1^2 + \cdots + x_n^2} \in \mathbb{R}_{>0}.$$

Definition 1.3. A vector product $\star$ on $\mathbb{R}^n$ is **norm-preserving** if the norm of the product is the product of the norms:

$$|\mathbf{x} \star \mathbf{y}| = |\mathbf{x}| \cdot |\mathbf{y}|.$$

Definition 1.4. $\star$ is **unital** if there is a vector $\mathbf{1} \in \mathbb{R}^n$ such that

$$\mathbf{1} \star \mathbf{x} = \mathbf{x} = \mathbf{x} \star \mathbf{1},$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Theorem 1.5 (Hurwitz, 1898)**.** *If $\star$ is a norm-preserving, unital vector product on $\mathbb{R}^n$ then $n = 1$, 2, 4 or 8, and in some orthonormal basis for $\mathbb{R}^n$, $\star$ is*

- *the ordinary product on $\mathbb{R}$;*
- *the complex product on $\mathbb{R}^2$;*
- *the quaternion product on $\mathbb{R}^4$; or*
- *the octonion product on $\mathbb{R}^8$.*

Here are the specifications of these special products:

n	Name and notation	Standard basis	Products of basis vectors
1	real numbers $\mathbb{R}$	$1 = \mathbf{e}_1$	$1 \cdot 1 = 1$
2	complex numbers $\mathbb{C}$	$1 = \mathbf{e}_1$, $i = \mathbf{e}_2$,	$1 \cdot 1 = 1$, $1 \cdot i = i = i \cdot 1$, $i \cdot i = -1$
4	quaternions $\mathbb{H}$	$1 = \mathbf{e}_1$, $i = \mathbf{e}_2$, $j = \mathbf{e}_3$, $k = \mathbf{e}_4$	$1 \cdot 1 = 1$, $1 \cdot i = i = i \cdot 1$, $1 \cdot j = j = j \cdot 1$, $1 \cdot k = k = k \cdot 1$, $i \cdot i = j \cdot j = k \cdot k = -1$ $i \cdot j = k = -j \cdot i$, $j \cdot k = i = -k \cdot j$, $k \cdot i = j = -i \cdot k$
8	octonions $\mathbb{O}$		(it's complicated...)

Each of these products has its own special properties, and is worth studying in its own right. (In this course, we'll study the complex product.)

There is also an interesting vector product on $\mathbb{R}^3$, the cross product $\times$, which is not norm-preserving (in fact $\mathbf{x} \times \mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^3$). It is interesting for a different reason: it is *rotation invariant.* That is, if R is a rotation of $\mathbb{R}^3$ by some angle about some axis then $R(\mathbf{x}) \times R(\mathbf{y}) = R(\mathbf{x} \times \mathbf{y})$. This property makes it relevant to scientific applications, because physical laws should look the same no matter how we rotate our coordinate axes.

The cross product is related to the quaternion product. If we take our standard basis for $\mathbb{R}^3$ to be $\{i, j, k\}$, then the cross product is the vector product defined by

$$i \times i = j \times j = k \times k = \mathbf{0}, \quad i \cdot j = k = -j \cdot i, \quad j \cdot k = i = -k \cdot j, \quad k \cdot i = j = -i \cdot k.$$

We can write any quaternion as $(a, \mathbf{x})$ where $a = a1 \in \mathbb{R}$ and $\mathbf{x} = x_1 i + x_2 j + x_3 k \in \mathbb{R}^3$. Then in the quaternions we have

$$(0, \mathbf{x}) \cdot (0, \mathbf{y}) = (-\mathbf{x} \cdot \mathbf{y}, \mathbf{x} \times \mathbf{y}).$$

There is also an interesting vector product on $\mathbb{R}^7$, related to the octonions in the same way that the cross product is related to the quaternions.

1.2 The complex numbers

We now focus our attention on the norm-preserving vector product on $\mathbb{R}^2$.

Definition 1.6. The system of **complex numbers** $\mathbb{C}$ is the plane $\mathbb{R}^2$ equipped with the vector product given by

$$1 \cdot 1 = 1, \quad 1 \cdot i = i = i \cdot 1, \quad i \cdot i = -1,$$

where $1 = \mathbf{e}_1 = (1, 0)$ and $i = \mathbf{e}_2 = (0, 1)$.

We write a typical complex number as z. In more detail, we write it as $x1 + yi$ (but we drop the 1 and write $x + yi$). We denote the product by a dot or by juxtaposition, just as with real numbers. We have by the distributive law

$$\begin{aligned}(x_1 + y_1 i) \cdot (x_2 + y_2 i) &= x_1 x_2 (1^2) + y_1 y_2 (i^2) + x_1 y_2 (1 \cdot i) + x_2 y_1 (i \cdot 1) \\ &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i.\end{aligned}$$

For example,

$$(1 + i)^2 = (1 - 1) + 2i = 2i.$$

A historical remark. In the 18th and 19th Centuries, and also in substandard 20th Century texts, complex numbers were sometimes introduced as follows. *We know that the square of any real number is non-negative. In particular, there is no real number x such that $x^2 = -1$. Nevertheless, let's introduce an 'imaginary number' i whose defining property is that $i^2 = -1$. We'll go on doing algebra with sums $x + yi$ using the same rules that we use with the real numbers, except that $i^2 = -1$.*

With this 'definition', complex numbers appear to be fictitious. The texts promised that if we suspend our disbelief, we would find complex numbers useful in solving problems about real numbers.

From a modern perspective, there's no need to suspend disbelief. I told you exactly what i is: it's the vector $(0, 1) \in \mathbb{R}^2$. And I told you what i^2 means: it means that we use a certain vector product on $\mathbb{R}^2$. Complex numbers are a mathematical entity on just the same footing as the real numbers.

2 Lecture 2: algebra in $\mathbb{C}$

2.1 Real and imaginary parts

Definition 2.1. If $z = x + yi \in \mathbb{C}$, we call the real number x the **real part** of z and denote it $\mathsf{Re}\, z$. We call the real number y the **imaginary part** and denote it $\mathsf{Im}\, z$.

Warning: the imaginary part is y, not yi. For example, $\mathsf{Im}(7 + 9i) = 9$.

2.2 Algebraic properties of the complex product

Lemma 2.2. *The product on $\mathbb{C}$ is unital. In fact, $1z = z = z1$ for all $z \in \mathbb{C}$.*

Proof. Write $z = x + yi = x1 + yi$. We have $1z = 1(x1 + yi) = x(1)^2 + y(1i) = x1 + yi = z$. Similarly, $z = z1$. □

Lemma 2.3. *The complex product is commutative ($z_1 z_2 = z_2 z_1$) and associative $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.*

Sketch of proof. Use the distributive law to convince yourself that if these properties are true when z_1, z_2 and z_3 are all equal to 1 or i then they are true always. We have $1i = i1$: this gives commutativity. We have $1(1i) = i = (1^2)i$, and we can check seven more equations of the same kind. These give associativity. □

Remark. A more conceptual method of proof is to observe that $\mathbb{C}$ with its complex product can be identified with a structure that is manifestly commutative and associative. This is indeed possible: let $\mathbb{R}[x]$ denote the set of all formal polynomial expressions with real coefficients, $a_0 + a_1 x + \cdots + a_d x^d$. We can add and multiply polynomials in the 'obvious' way, and the multiplication is associative and commutative. Now let $\mathbb{R}[x]/(x^2 + 1)$ denote the same polynomials, now written $[a_0 + a_1 x + \cdots + a_d x^d]$, but now a polynomial of the form $f(x)(x^2 + 1)$ counts as 0: $[f(x)(x^2 + 1)] = [0]$. Add and multiply as before. We can identify $\mathbb{C}$ with $\mathbb{R}[x]/(x^2 + 1)$ by mapping $1 \in \mathbb{C}$ to the polynomial $[1]$, and i to $[x]$. Note that $[x^2] = [(x^2 + 1)] + [-1] = [0] + [-1] = [-1]$.

Let's look at $\mathbb{C}$ with the structure we know about (most of it, anyway —I mentioned a norm-preserving property last time, and we'll return to that soon). There's addition and multiplication, and we have special complex numbers 0 and 1. In fact, $\mathbb{C}$ is an example of a *commutative algebra* over $\mathbb{R}$. That means

- We have a copy of $\mathbb{R}$ inside $\mathbb{C}$ (namely, complex numbers of form $x + 0i$). This copy of $\mathbb{R}$ is closed under addition and multiplication[1], and these structures are the usual ones for $\mathbb{R}$.
- Addition in $\mathbb{C}$ is associative and commutative: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, and $z_1 + z_2 = z_2 + z_1$.
- 0 is neutral for addition: $z + 0 = z$.

[1] Not true of the imaginary line $i\mathbb{R}$.

- Multiplication is associative and commutative: $(z_1z_2)z_3 = z_1(z_2z_3)$
- 1 is neutral for multiplication: $1z = z$.
- $0z = 0$.
- Addition distributes over multiplication.

Other algebraic properties follow from these, e.g. $(-1)z + z = 0$.

2.3 Complex conjugation

Definition 2.4. The **conjugate** of the complex number $z = x + yi$ is the complex number $\bar{z} = x - yi$. *Complex conjugation* is the function $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto \bar{z}$.

Complex conjugation $\mathbb{C} \to \mathbb{C}$ is a sensible thing to write down because it's a *symmetry of* $\mathbb{C}$ *as a real commutative algebra*. That is, it respects addition and multiplication, 0 and 1, and it does nothing to the copy of $\mathbb{R}$ inside $\mathbb{C}$.

- $\boxed{\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1z_2} = \bar{z}_1\bar{z}_2}$
- $\bar{1} = \bar{1}, \quad \bar{0} = 0.$
- If $x \in \mathbb{R} \subset \mathbb{C}$ then $\bar{x} = x$.

These are easy to check. For instance,

$$\begin{aligned}\overline{z_1z_2} = \overline{(x_1 + y_1i)(x_2 + y_2i)} &= \overline{(x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i} \\ &= (x_1x_2 - y_1y_2) - (x_1y_2 + x_2y_1)i \\ &= (x_1 - y_1i)(x_2 - y_2i) \\ &= \bar{z}_1\bar{z}_2.\end{aligned}$$

Notice also that

$$\mathsf{Re}\, z = \frac{1}{2}(z + \bar{z}), \quad \mathsf{Im}\, z = -\frac{i}{2}(z - \bar{z}).$$

Now we come to an absolutely crucial property of complex conjugation—one you must learn to use with great frequency! First a definition:

Definition 2.5. For $z = x + yi$, write $|z| = \sqrt{x^2 + y^2} \in \mathbb{R}_{\geq 0}$. We call $|z|$ the **norm** or **modulus** (two other names are **length** or **magnitude**) of z.

Geometrically, if we think of z as a point $(x, y) \in \mathbb{R}^2$, $|z|$ is its distance from the origin.

Here's the key property:

$$\boxed{z\bar{z} = |z|^2}$$

Indeed, $z\bar{z} = (x + yi)(x - yi) = x^2 + y^2 = |z|^2$.

I'll call this the *z-z-bar identity.* Here's a first example of the z-z-bar identity in use:

Lemma 2.6. *The product on $\mathbb{C}$ is norm-preserving: $|zw| = |z|\,|w|$.*

Proof. We have

$$|zw|^2 = (zw)\overline{zw} = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = |z|^2|w|^2.$$

Taking non-negative square roots we deduce that $|zw| = |z|\,|w|$. □

2.4 Reciprocals

In a real commutative algebra, a *reciprocal* for an element z is an element z^{-1} such that $z^{-1}z = 1$. Only non-zero elements can have reciprocals, since $w0 = 0$ for any w. In general, z might not have a reciprocal, but it if has one it's *unique*: say $wz = 1$ and $w'z = 1$; then $w = 1w = (w'z)w = w'(zw) = w'1 = w'(wz) = w'1 = w'$.

Proposition 2.7. *If $z = x + yi \in \mathbb{C}$, $z \neq 0$, then the complex number*

$$\frac{1}{|z|^2}\bar{z} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i$$

is a reciprocal for z (we denote it by z^{-1} or $1/z$).

Proof. By the z-z-bar identity, we have

$$\frac{1}{|z|^2}\bar{z}z = \frac{1}{|z|^2}|z|^2 = 1.$$

□

A real commutative algebra in which every non-zero element has a reciprocal is called a *field*, or more precisely, an *extension field of* $\mathbb{R}$ to emphasize that it contains the real numbers. So $\mathbb{C}$ is an extension field of $\mathbb{R}$.

Example 2.8. We have $i^{-1} = \bar{i}/|i|^2 = (-i)/1 = -i$.

Example 2.9. Let's compute $(1 + i)^{-1}$. We have $|1 + i|^2 = 2$, so $(1 + i)^{-1} = (1 - i)/2 = (1/2) - (1/2)i$.

Example 2.10. Let's compute $(z + i)^{-1}$, assuming $z = x + yi \neq i$. We have $|z + i|^2 = (z + i)(\overline{z + i}) = (z + i)(\bar{z} + \bar{i}) = (z + i)(\bar{z} - i) = z\bar{z} - i(z - \bar{z}) + 1 = x^2 + y^2 - 2y + 1$. So

$$\frac{1}{z + i} = \frac{\bar{z} - i}{z\bar{z} - i(z - \bar{z}) + 1} = \frac{x}{x^2 + y^2 - 2y + 1} + \frac{-y - 1}{x^2 + y^2 - 2y + 1}i.$$

Advice when doing algebra with complex numbers. It is possible but inadvisable to do calculations such as the last one by immediately writing $z = x + iy$ and working with x and y. By doing so, you lose all the power of the complex number system. It's much better to work with z itself until just before the end. In this example, the methods aren't too different, but as we continue, the difference between the methods will become large.

2.5 Ordering

There is one important structure which is available in $\mathbb{R}$ but not in $\mathbb{C}$. Namely, in $\mathbb{R}$ we have a notion of inequality $<$, and for any x, y in $\mathbb{R}$, exactly one of three options holds: $x < y$; $y < x$; or $x = y$. Moreover, if $x_1 < x_2$ and $x_2 < x_3$ then $x_1 < x_3$. Such a notion $<$ is called a *total order*. It's possible (exercise!) to find a total order on $\mathbb{C}$.

But it turns out that any such order will behave badly from an algebraic point of view. In $\mathbb{R}$, $<$ is 'compatible with addition and multiplication':

(i) if $x < y$ then $x + z < y + z$.

(ii) If $0 < x$ and $y < z$ then $xy < xz$.

If we ask for an ordering of $\mathbb{C}$ which is compatible with addition and multiplication, we run into a contradiction: since $i \neq 0$, we must have either $0 < i$ or $i < 0$. Say $0 < i$. Then by (ii) we have $0^2 < i^2$, i.e., $0 < -1$. This implies that $0 < 1$ (using (ii)) and that $1 < 0$ (using (i)), but only one of these two possibilities is allowed—a contradiction. If $0 < i$ we find similar trouble.

Upshot: there isn't an algebraically-sensible ordering of $\mathbb{C}$. Don't write $z < w$ for complex numbers z and w, unless they happen to be real.

3 Lecture 3: Geometry of $\mathbb{C}$

Warning: these notes are woefully lacking in pictures. Almost everything in them should have an accompanying picture!

A complex number $z = x + yi$ has cartesian coordinates $(x, y) \in \mathbb{R}^2$. Its distance from the origin is $|z| = \sqrt{x^2 + y^2} \geq 0$. For z and w in $\mathbb{C}$, the real number $|z - w|$ represents the distance between them.

Example 3.1. The solutions to the equation

$$|z - i| = 2$$

are those complex numbers z whose distance from i is 2 (a *circle*). We can write out this equation as

$$4 = |z - i|^2 = (z - i)\overline{(z - i)} = z\bar{z} - 1 + i(z - \bar{z}) = (x^2 + y^2 + 1) - 2y,$$

i.e., $x^2 + y^2 - 2y = 3$, or $x^2 + (y - 1)^2 = 4$, which would be the more familiar way to represent this circle.

Example 3.2. *Problem.* Identify geometrically the locus of complex numbers z whose distance from 0 is 3 times their distance from i. *Solution:* these are the solutions to

$$|z - i| = 3|z|.$$

Squaring and using the z-z-bar identity, we get the equivalent (why?) equation

$$z\bar{z} - i\bar{z} + iz + 1 = 9z\bar{z},$$

i.e.,

$$z\bar{z} + \frac{i}{8}(\bar{z} - z) - \frac{1}{8} = 0.$$

Now 'complete the square', writing this equation as

$$\left|z + \frac{i}{8}\right|^2 - \frac{1}{8} - \frac{1}{64} = 0,$$

i.e.

$$\left|z + \frac{i}{8}\right|^2 = \frac{9}{64}$$

which represents a circle, center $-i/8$, radius $3/8$.

3.1 Polar coordinates

Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. On $\mathbb{C}^*$ we have polar coordinates (r, θ), where $r \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R} \mod 2\pi\mathbb{Z}$. Notice that $r = |z|$, the modulus of z. The angle θ is called the *argument* of z, and denoted $\mathsf{arg}\, z$. It is measured counterclockwise from the real, positive axis.

Note: By $\mathbb{Z}$, I denote the integers. The notation $\mathbb{R} \mod 2\pi\mathbb{Z}$ means the system of real numbers, modified so that two real numbers are considered identical if they differ by an integer multiple of 2π.

In the following examples, the argument is evident from a picture:

Example 3.3. $\mathsf{arg}\, i = [\pi/2]$ (by the square brackets, I emphasize that we are working modulo 2π; we also have $\mathsf{arg}\, i = [-246\pi + \pi/2]$, for instance).

Example 3.4. $\mathsf{arg}(-1) = [\pi]$.

Example 3.5. $\mathsf{arg}(1 - i) = [3\pi/4]$.

Example 3.6. We have $\mathsf{arg}(\bar{z}) = -\mathsf{arg}(z)$ and $|\bar{z}| = |z|$. (Complex conjugation is reflection in the real axis.)

The relation between the cartesian and polar coordinates on $\mathbb{C}^*$ is as follows (let $z = z + yi$):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

In the other direction,

$$r = |z| = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x \in [-\infty, \infty].$$

Warning: If $\arctan(a)$ denotes the unique solution ϕ to $\tan\phi = a$ with $\phi \in (-\pi/2, \pi/2)$ and $x \geq 0$ then we have $\theta = \arctan(y/x)$. But if $x < 0$ this can't be right. I recommend that you find θ by drawing a picture and thinking it through, not by memorizing a formula. What is definitely true is that if you know $r\cos\theta$ and $r\sin\theta$ (for $r > 0$) then θ is uniquely determined.

So: for $z \in \mathbb{C}^*$, we have

$$z = x + yi = r(\cos\theta + i\sin\theta).$$

3.2 De Moivre's theorem

We know that $|z_1||z_2| = |z_1 z_2|$: that is, the *modulus multiplies* when you multiply complex numbers.

Theorem 3.7 ('argument adds'). *For z_1, $z_2 \in \mathbb{C}^*$ we have*

$$\mathsf{arg}(z_1 z_2) = \mathsf{arg}(z_1) + \mathsf{arg}(z_2).$$

Proof. Write $r_1 = |z_1|$, $r_2 = |z_2|$, $[\theta_1] = \mathsf{arg}\, z_1$, $[\theta_2] = \mathsf{arg}\, z_2$. Then

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right] \\ &= r_1 r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right], \end{aligned}$$

using the angle-addition formulae for sine and cosine. Hence $[\theta_1 + \theta_2] = \mathsf{arg}(z_1 z_2)$. □

In particular, if $|z| = 1$ then *multiplying a complex number by z rotates it counterclockwise by* $\mathsf{arg}\, z$. For instance, *multiplication by i is rotation by* $\pi/2$.

If we don't assume $|z| = 1$, then multiplying by z means you first rotate by $\mathsf{arg}\, z$, then you scale by $|z|$.

Corollary 3.8 (de Moivre's theorem). *For each $n \in \mathbb{Z}$, we have*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Proof. Case 1: $n > 0$. We reach $(\cos\theta + i\sin\theta)^n$ by starting at 1 and then rotating by θ, n times. So we rotate overall by $n\theta$. Thus $(\cos\theta + i\sin\theta)^n.1 = (\cos n\theta + i\sin n\theta).1$.

Case 2: $n = 0$. We define $z^0 = 1$ for each $z \in \mathbb{C}^*$. So when $n = 0$, de Moivre's theorem says $1 = \cos 0 + i\sin 0$, which is correct.

Case 3: $n < 0$. Write $n = -p$. By definition,

$$(\cos\theta + i\sin\theta)^n = \left[(\cos\theta + i\sin\theta)^p\right]^{-1},$$

which by case 1 is

$$\left[\cos p\theta + i\sin p\theta\right]^{-1}.$$

This is the reciprocal of a complex number z with $|z| = 1$, so $z^{-1} = \bar{z}$, i.e.,

$$\left[\cos p\theta + i\sin p\theta\right]^{-1} = \cos p\theta - i\sin p\theta = \cos n\theta + i\sin n\theta.$$

So the formula checks. □

3.3 Roots

We can use the 'argument adds' theorem to solve the equation

$$z^n = 1$$

for each positive integer n. For any solution z, we must have $|z|^n = 1$, so $|z| = 1$; and we must have $[n \operatorname{arg} z] = [0]$, i.e., $n \operatorname{arg} z = 2\pi k$ for some $k \in \mathbb{Z}$. So $\operatorname{arg} z = 2\pi k/n$. Modulo 2π, the possible arguments are

$$0,\ 2\pi/n,\ 3\pi/n, \ldots 2\pi(n-1)/n.$$

So the n solutions are

$$\cos\frac{2\pi j}{n} + i\sin\frac{2\pi j}{n}, \quad j = 0, \ldots, n.$$

Example 3.9. The cube roots of 1 are 1, $\cos 2\pi/3 + i\sin 2\pi/3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\cos 4\pi/3 + i\sin 4\pi/3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

More generally, we can solve

$$z^n = w$$

where w is given. Take $\theta \in \mathbb{R}$ with $[\theta] = \operatorname{arg} w$. One solution is

$$z = |w|^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right).$$

If z is a solution and $\zeta^n = 1$ then $z\zeta$ is another solution. So we have n solutions (all different unless $w = 0$) given by

$$|w|^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)\left(\cos\frac{2\pi j}{n} + i\sin\frac{2\pi j}{n}\right), \quad j = 0, \ldots, n.$$

A simple case is that of square roots: the solutions of $z^2 = w$ are given by

$$\pm|w|^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right).$$

Example 3.10. For any complex numbers $a \neq 0$, b and c, we can solve the quadratic equation $az^2 + bz + c = 0$. We complete the square:

$$az^2+bz+c = a\left(z^2 + \frac{b}{a}z + \frac{c}{a}\right) = a\left((z + \frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right) = a\left((z + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a^2}\right),$$

so $az^2 + bz + c = 0$ iff $(z + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$. Knowing how to take complex square roots, we arrive at the usual quadratic formula,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Theorem 3.11 ('fundamental theorem of algebra')**.** *Take a polynomial* $p(z) = a_d z^d + a_{d-1} z^d + \cdots + a_0$*, where* $d > 0$ *and* $a_d \neq 0$ *(i.e.,* p *has positive degree* d*). The coefficients* a_j *are complex. Then there is a complex number* z_1 *such that* $p(z_1) = 0$*.*

We'll prove the theorem later in the course. We can factor out $z - z_1$, writing $p(z) = (z - z_1)q(z)$ where q is a polynomial of degree $d - 1$. If $d - 1 > 0$ we can then find a root of q, and repeat the process. So we deduce that

$$p(z) = a_z(z - z_1)(z - z_2) \cdots (z - z_d).$$

Corollary 3.12 (Gauss)**.** *Let* $f(x) = a_d x^d + \cdots + a_0$ *where* $a_j \in \mathbb{R}$*,* $a_d \neq 0$ *and* $d > 0$*. Then* $f = g \cdot h$*, where* g *and* h *are real polynomials and* g *is either linear (*$g(x) = x - x_1$*) or irreducible quadratic (*$g(x) = x^2 + bx + c$ *with* $b^2 - 4c < 0$*).*

Proof. Regard f as a complex polynomial function $f(z)$ whose coefficients happen to be real. By FTA, it has a complex root z_1. If z_1 is real, we're done. Notice that $\bar{z}_1$ is again a root, since

$$0 = \overline{f(z_1)} = \overline{a_d z_1^d + \cdots + a_0} = \overline{a}_d \overline{z}_1^d + \cdots + \overline{a}_0 = a_d \overline{z}_1^d + \cdots + a_0.$$

So if z_1 is not real then $(z - z_1)(z - \bar{z}_1)$ is a factor of $f(z)$. We have

$$(z - z_1)(z - \bar{z}_1) = z^2 - 2\,\mathsf{Re}(z_1)z + |z_1|^2.$$

Writing $b = -2\,\mathsf{Re}\,z_1$ and $c = |z_1|^2$, we have $b^2 - 4c = 4((\mathsf{Re}\,z_1)^2 - |z_1|^2) = -4(\mathsf{Im}\,z_1)^2 < 0$. □