M361 Theory of functions of a complex variable

T. Perutz

U.T. Austin, Fall 2012

Lecture 4: Exponentials and logarithms

We have already been making free use of the sine and cosine functions,

$$\cos\colon \mathbb{R} \to \mathbb{R}, \quad \sin\colon \mathbb{R} \to \mathbb{R}.$$

Their geometric meanings are familiar. In a rigorous mathematical development in which analysis precedes geometry, one might choose to define them by their power series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4^1} - \cdots = \sum_{n\geq 0} (-1)^n \frac{x^{2n}}{(2n)!},$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

In any event, it's useful to prove (using the ratio test, say) that these series *converge absolutely* on $\mathbb{R}$. This justifies term-by-term differentiation, which shows that $d(\sin x)/dx = \cos x$ and $d(\cos x)/dx = -\sin x$.

The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ can also be defined through its power series, which is again absolutely convergent by the ratio test:

$$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n\geq 0} \frac{x^n}{n!}.$$

One has

$$\exp(x+y) = \exp(x)\exp(y),$$

as one can prove by using absolute convergence to justify a term-by-term multiplication of the two series on the right.

Euler carried out the following thought-experiment: what happens if we *formally* compute $\exp(i\theta)$ using the exponential series? By formally, I mean that we just move about our symbols in a reasonable-looking way, without yet knowing exactly what it means. In that spirit, we have

$$\begin{aligned}\exp(i\theta) &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \dots\right) \\ &= \cos\theta + i\sin\theta.\end{aligned}$$

Because of this manipulation, the following definition is not crazy:

Definition 4.1. For $\theta \in \mathbb{R}$, we define $\exp(i\theta) = \cos\theta + i\sin\theta \in \mathbb{C}$. For $z = x + yi \in \mathbb{C}$, we define $\exp(z) = \exp(x)\exp(yi) = e^x(\cos y + i\sin y)$.

So $\exp(i\theta)$ lies on the unit circle, and $\mathsf{arg}\exp(i\theta) = [\theta]$, while $\exp(x+iy)$ lies inside the unit circle if $x < 0$ on it if $x = 0$, and outside it if $x > 0$; its argument is $[y]$.

Example 4.2. A famous equation of Euler's is

$$\exp(i\pi) + 1 = 0,$$

which holds because $\exp(i\pi) = \cos\pi + i\sin\pi = -1$. We see in it the basic constants 0, 1, π and i, and the basic operations of addition, multiplication and exponentiation.

Proposition 4.3. $\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$ *for all* $z_1, z_2 \in \mathbb{C}$.

Proof. When z_1 and z_2 are pure imaginary, this follows from the definition and the 'argument adds' theorem. When they are purely real, it follows from the corresponding fact about real exponentials. The general case follows from the real and imaginary cases. □

We often write e^z instead of $\exp z$. Note that the polar form $r(\cos\theta + i\sin\theta)$ can be rewritten as $re^{i\theta}$.

4.1 Complex sine and cosine

We have, for $\theta \in \mathbb{R}$, $e^{i\theta} = \cos\theta + i\sin\theta$, and $e^{-i\theta} = \cos\theta - i\sin\theta$. Rearranging, we get

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-\theta}), \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

Definition 4.4. For $z \in \mathbb{C}$, we define

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz} - e^{iz}).$$

This definition agrees with our old one when $z \in \mathbb{R} \subset \mathbb{C}$. Note that for $y \in \mathbb{R}$,

$$\cos(iy) = \cosh y, \quad \sin(iy) = i\sinh y.$$

The following lemma says that a familiar property of (real) sine and cosine persists with the extended definition:

Lemma 4.5. *We have* $\cos^2 z + \sin^2 z = 1$.

Proof. We compute

$$4(\cos^2 z + \sin^2 z) = (e^{iz} + e^{-iz})^2 - (e^{iz} - e^{-iz})^2 = e^{iz}e^{-iz} = 4e^0 = 4.$$

□

4.2 Logarithms

Let's try to solve $\exp(z) = w$, given $w \in \mathbb{C}$. Writing $z = x + yi$, we have $e^z = e^x e^{iy}$. Let $w = re^{i\phi}$. Then $\exp(z) = w$ is equivalent to

$$r = e^x, \quad [y] = [\phi] \in \mathbb{R}/2\pi\mathbb{Z}.$$

The first equation has no solution when $w = 0$; when $w \neq 0$, it has the unique solution, $x = \log r$. The second equation has solutions

$$y = \phi + 2\pi ik, \quad k \in \mathbb{Z}.$$

So there are infinitely many possibilities for $\mathsf{Im}\, z$, which differ by integer multiples of 2π.

We would like to define $\log w$, for $w \neq 0$, to mean the solution to $\exp(z) = w$, but we note that log is a multi-valued function, since there are infinitely many solutions. The formula is

$$\log w = \log |w| + i \,\mathsf{arg}(w).$$

Example 4.6. Let's compute $\log i$. We have $\log i = \log 1 + i \,\mathsf{arg}\, i = 0 + i\pi \left(\frac{1}{2} + 2k\right)$, where $k \in \mathbb{Z}$ is arbitrary.

We can try to improve the situation by making a continuous *choice* of argument from among the possibilities. We could *demand* that $\mathsf{arg}\, w \in [0, 2\pi)$, and then define a single-valued logarithm by $\log w = \log |w| + i \,\mathsf{arg}\, w$. This is called the *principal branch* of the logarithm. It's useful, but it has an unsettling discontinuity. Namely, suppose we go around the circle, letting θ run from 0 to 2π. We have

$$\log e^{i\theta} = i\theta$$

for $\theta \in [0, 2\pi)$, so one might expect that $\log e^{i2\pi}$ should be 2π, but actually $\log e^{i2\pi} = 0$, which is a jump.

If we make a *branch cut* by deleting from $\mathbb{C}$ the non-negative real axis, we get a perfectly well-behaved logarithm function.

4.3 Complex exponents

Can we make sense of complex powers z^w? *Yes*: note that when $a > 0$ and b are real, we have $a^b = e^{b \log a}$. We can use this to extend the definition:

Definition 4.7. For $w \in \mathbb{C}$, define a *multi-valued* function $\mathbb{C}^* \ni z \mapsto z^w$ by

$$z^w = e^{w \log z}.$$

Here we must use all possible values of $\log z$.

Example 4.8. Take $w = 1/n$, where n is a positive integer. Let $z = re^{i\theta}$. We have

$$z^{1/n} = e^{(\log z)/n} = e^{(\log r)/n} e^{i(\theta+2\pi k)/n} = e^{(\log r)/n} e^{i\theta/n} (e^{2\pi i/n})^k,$$

where $k \in \mathbb{Z}$ is arbitrary. Since $(e^{2\pi i/n})^n = 1$, there are n possible values for $z^{1/n}$.

On the cut plane where we delete the positive real axis, we can consistently define a single valued nth root function by

$$z^{1/n} = e^{(\log r)/n} e^{i\theta/n}, \quad \theta \in (0, 2\pi).$$

Example 4.9. Let's compute z^i for $i \in \mathbb{C}^*$. We have $z^i = e^{i \log z} = e^{-\mathsf{arg}(z)+i\log|z|} = e^{-\mathsf{arg}(z)} e^{i \log|z|}$. Notice that in this case the argument, $\log|z|$, is unambiguous, while the modulus can be rescaled by $e^{2\pi k}$ for any $k \in \mathbb{Z}$.

Lecture 5: Sequences and limits

For $c \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$, let's denote by $D(c; r)$ the open disc (i.e., no boundary), centered at $c \in \mathbb{C}$, with radius r:

$$D(c; r) = \{z \in \mathbb{C} : |z - c| < r\}.$$

Definition 5.1. (i) A *sequence* (a_n) in $\mathbb{C}$ is a function $\mathbb{N} \to \mathbb{C}$ from the natural numbers to $\mathbb{C}$, $n \mapsto a_n$.

(ii) A sequence (a_n) *converges* to $a \in \mathbb{C}$ if for any $\epsilon > 0$, however small, the sequence eventually stays within the ϵ-disc $D(a; \epsilon)$. In this case we write $a_n \to a$ or $\lim_{n\to\infty} a_n = a$, and say that a is the *limit* of the sequence.

By 'eventually', I mean that, for each ϵ, there's some $N = N(\epsilon)$ such that for all $n \geq N$ we have $a_n \in D(a; \epsilon)$.

Definition 5.2. We say that the sequence *diverges to* ∞ and write $a_n \to \infty$ if for any $R > 0$, however large, the sequence eventually stays *out* of the disc $D(0; R)$. That is, for each R there's an $N = N(R)$ such that for all $n \geq N$ we have $|a_n| \geq R$.

Example 5.3. Fix $z \in \mathbb{C}$, and let $a_n = z^n$.

- If $|z| < 1$ then $a_n \to 0$. Indeed, we have $|z^n| = |z|^n$. For any $\epsilon > 0$, there's an N such that whenever $n \geq N$ we have $|z|^n < \epsilon$ (take $n = \log\epsilon / \log|z|$ if $z \neq 0$, or $n = 1$ if $z = 0$—why does this work?). So $z_n \in D(0; \epsilon)$.

- If $|z| > 1$ then $a_n \to \infty$.
- If $|z| = 1$ with $z = 1$ then a_n has no limit. Indeed, $|a_{n+1} - a_n| = |z|^n|z-1| = |z-1|$. If we take $\epsilon < |z-1|/2$ then a_{n+1} and a_n don't both belong to the same ϵ-disc, whatever its center. This is because of the *triangle inequality.*

5.1 The triangle inequality

The triangle inequality says that

$$|z + w| \leq |z| + |w|.$$

Geometrically, this is the obvious statement that the length of one side of a triangle is at most the sum of the lengths of the other two sides. Algebraically, we can prove it as follows:

Proof. We have

$$\frac{1}{2}(|z+w|^2 - (|z| + |w|)^2) = \frac{1}{2}(z\bar{w} + \bar{z}w) - |z||w| = \mathsf{Re}(z\bar{w}) - |z||w|.$$

Now a short computation shows that $\mathsf{Re}(z\bar{w})$ is the dot product of z and w viewed as vectors. So

$$\mathsf{Re}(z\bar{w}) = |z||w|\cos\phi,$$

where ϕ is the angle between the vectors. So $\mathsf{Re}\, z\bar{w} \leq |z||w|$, and hence

$$|z+w|^2 - (|z| + |w|)^2 \leq 0.$$

□

The triangle inequality has consequences which are just as useful:

- $|z_1 - z_3| \leq |z_1 - z_2| + |z_2 - z_3|$.
  (Just apply TI with $z = z_1 - z_2$ and $z = z_2 - z_3$).
- $|z_1 - z_2| \geq |z_1| - |z_2|$.
  (Apply TI to $z = z_1 - z_2$ and $w = z_2$.)

5.2 The Mandelbrot set

Definition 5.4. A sequence (a_n) is *bounded* if there's some disc $D(0; R)$ which contains all the a_n. That is, there's an $R > 0$ such that $|z_n| < R$ for all n.

For any $c \in \mathbb{C}$, define a sequence $(z_n) = (z_n(c))$ as follows. Put $z_0 = 0$. Recursively assume that $z_0, z_1, \ldots, z_n$ have been defined, and put

$$z_{n+1} = z_n^2 + c.$$

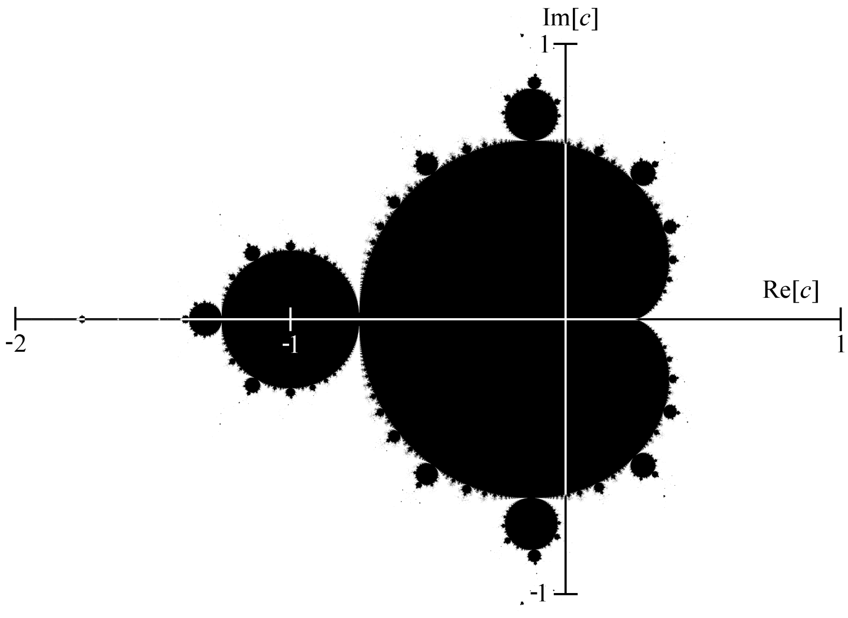

Figure 1:

So if $c = 1$ we get the sequence

$$0, \quad 1, \quad 2, \quad 5, \quad 26, \quad \ldots.$$

If $c = i$ we get

$$0, \quad i, \quad -1+i, \quad -i, \quad -1+i, \quad -i, \quad \ldots.$$

Definition 5.5. The *Mandelbrot set* M is the set of $c \in \mathbb{C}$ for which the sequence $(z_n(c))$ is bounded.

Figure 1 shows the Mandelbrot set is shown (in black). As you can see, there is a large heart-shaped region (the 'cardioid'), numerous circular 'bulbs' sprouting from it, and then a very intricate boundary—the intricacies persist when you zoom in.

Lemma 5.6. *M is contained in the closed disc* $\overline{D(0;2)} = \{z \in \mathbb{C} : |z| \leq 2\}$.

Proof. Suppose $|c| > 2$. I claim that $z_n(c) \to \infty$ (hence it's an unbounded sequence). Let $r = |c| - 1 > 1$. I claim that

$$|z_n(c)| \geq r_n|c|,$$

where the sequence r_n is defined recursively for $n \geq 2$ by $r_2 = r$ and $r_{n+1} = 2r_n^2 - 1$. Assume inductively that the inequality is true for n; then

$$\begin{aligned} |z_{n+1}(c)| &= |z_n(c)^2 + c| \\ &\geq |z_n(c)^2| - |c| && \text{triangle inequality} \\ &= |z_n(c)|^2 - |c| \\ &\geq (r_n^2|c| - 1)|c| && \text{inductive assumption} \\ &> (2r_n^2 - 1)|c| && \text{since } |c| > 2 \\ &= (r_{n+1})|c| \end{aligned}$$

as claimed. For the base step $n = 2$, a similar argument gives that $|z_2(c)| = |c^2 + c| \geq |c|^2 - |c| = |c|r_2$. Now write $r = 1 + \epsilon$, and observe, by induction again, that $r_n \geq 1 + 4^{n-2}\epsilon$ (indeed, this is obvious when $n = 2$, and if it's true for n then $r_{n+1} = 2r_n^2 - 1 \geq 2(1 + 2^{n-1}\epsilon)^2 - 1 \geq 1 + 4.4^{n-2}\epsilon$). Hence $r_n \to \infty$ as $n \to \infty$. This proves that $z_n(c) \to \infty$. □

Lemma 5.7. *Suppose that, for some c, there's an N such that $|z_N(c)| > 2$. Then $z_n(c) \to \infty$.*

Proof. If $|c| > 2$ then the result holds by the last lemma. So we may assume that $|c| \leq 2$. Say $|z_N(c)| = 2 + \epsilon$, with $\epsilon > 0$. Then

$$|z_{N+1}(c)| \geq |z_N(c)|^2 - |c| \geq |z_N(c)|^2 - 2 \geq (2 + \epsilon)^2 - 2 \geq 2 + 2\epsilon.$$

Check inductively that $|z_{N+k}(c)| \geq 2 + 2^k\epsilon$, which shows that $z_n(c) \to \infty$. □

This lemma has a practical consequence for making computer pictures of the Mandelbrot set. If for some c we find an iterate $z_N(c)$ of modulus > 2, we know that $c \notin M$ (assuming our calculation was accurate—and numerical errors *could* be a problem, especially near the boundary of M). In practice, to get a picture of M, what you do is scan through a grid of possible c values. You decide that you will compute the sequence $z_n(c)$ for $n \leq m$, where m is, say, 1000 (the larger the better, but the program will slow down). Also decide on a target radius $T > 2$. If you find $|z_N(c)| \geq T$ then make a note of the 'escape time' N and STOP because $c \notin M$. Otherwise, if $|z_m(c)| \leq T$ then you guess that $c \in M$ and plot c in black (but you might be wrong, and so the picture only has a certain accuracy).

Instead of plotting points $c \notin M$ white, you can plot different escape times in different colors, resulting in a more visually striking image, such as Figure 2 which shows a tiny region of the complex plane. If you take T large, you'll get finer color-gradations.

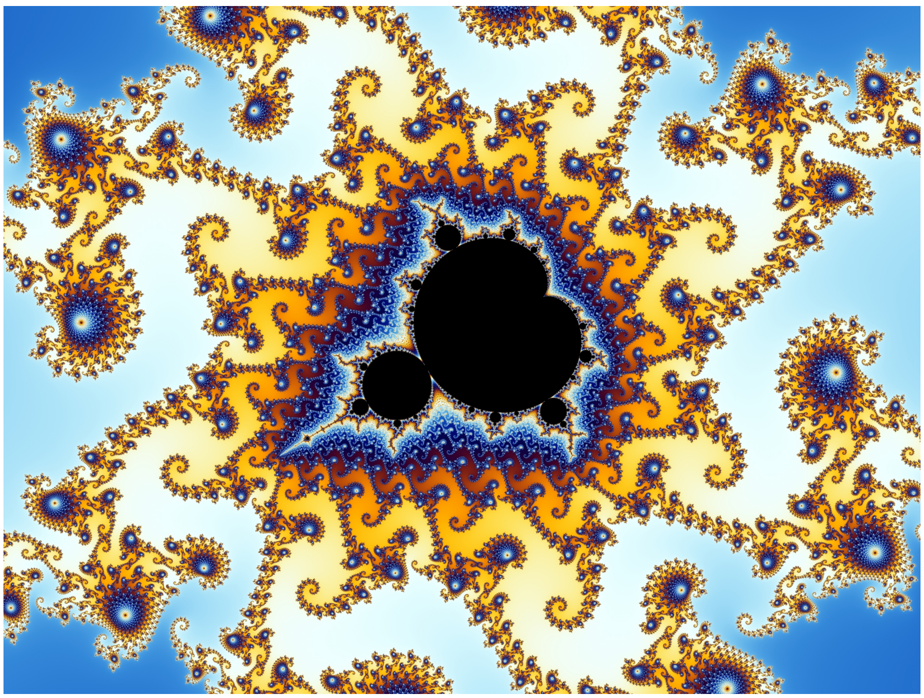

Figure 2: