Lecture 3. The intersection form

1 The cup product in middle dimensional cohomology

Suppose that M is a closed, oriented manifold of even dimension $2n$. Its middle-degree cohomology group $H^n(M)$ then carries a bilinear form, the *cup-product form*,

$$H^n(M) \times H^n(M) \to \mathbb{Z}, \quad (x, y) \mapsto x \cdot y := \mathsf{eval}\,(x \cup y, [M]).$$

The cup product $x \cup y$ lies in $H^{2n}(M)$, and eval denotes the evaluation of a cohomology class on a homology class.

The cup-product form is skew-symmetric when n is odd, and symmetric when n is even.

Lemma 1.1 *$x \cdot y$ is equal to the evaluation of x on $D_X y$.*

Proof Under the isomorphism $H_0(X) \cong \mathbb{Z}$ sending the homology class of a point to 1, one has $x \cdot y = (x \cup y) \cap [X] = x \cap (y \cap [X]) = \langle x, D_X y \rangle$. □

As we saw last time, when interpreted as a pairing on homology, $H_n(M) \times H_n(M) \to \mathbb{Z}$, by applying Poincaré duality to both factors, the cup product is an intersection product. Concretely, if S and S' are closed, oriented submanifolds of M, of dimension n and intersecting transversely, and $s = D^M[S]$, $s' = D^M[S']$, then

$$s \cdot s' = \sum_{x \in S \cap S'} \varepsilon_x,$$

where $\varepsilon_x = 1$ if, given oriented bases $(e_1, \ldots, e_n)$ of $T_x S$ and $(e'_1, \ldots, e'_n)$ of $T_x S'$, the basis $(e_1, \ldots, e_n, e'_1, \ldots, e'_n)$ for $T_x M$ is also oriented; otherwise, $\varepsilon_x = -1$.

Notation: For an abelian group A, let A' denotes its torsion-free quotient A/A_{tors}.

The cup product form necessarily descends to a form on the free abelian group $H^n(M)'$. We shall denote the latter form by Q_M.

Proposition 1.2 *The cup-product form Q_M is non-degenerate, i.e., the group homomorphism*

$$H^n(M)' \to \mathsf{Hom}(H^n(M'), \mathbb{Z}), \quad x \mapsto (y \mapsto x \cdot y)$$

is an isomorphism.

Proof By the lemma, an equivalent assertion is that evaluation defines a non-degenerate pairing of the torsion-free quotients $H^n(X)'$ and $H_n(X)'$. This is true as a matter of homological algebra: it is a weak form of the cohomological universal coefficients theorem. □

If we choose an integral basis $(e_1, \ldots, e_b)$ of $H^n(M)'$, we obtain a square matrix Q of size $b \times b$, where $b = b_n(M)$, with entries $Q_{ij} = e_i \cdot e_j$. It is symmetric or skew symmetric depending on the parity of n.

Exercise 1.3 Non-degeneracy of the form Q_M is equivalent to the unimodularity condition $\det Q = \pm 1$.

Proposition 1.4 *Suppose that N is a compact, oriented manifold with boundary M, and $i\colon M \to N$ the inclusion. Let $L = \operatorname{im} i^* \subset H^n(M;\mathbb{R})$. Then (i) L is isotropic, i.e., $x \cdot y = 0$ for $x, y \in L$; and (ii) $\dim L = \frac{1}{2}\dim H^n(M;\mathbb{R})$.*

Proof (i) We have $i^*u \cdot i^*v = \mathsf{eval}(i^*(u \cup v), [M]) = \mathsf{eval}(u \cup v, i_*[M])$. But $i_*[M] = 0$, since the fundamental cycle of M is bounded by that of N.

(ii) There is a commutative diagram with exact rows as follows:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^n(N;\mathbb{R}) & \xrightarrow{i^*} & H^n(M;\mathbb{R}) & \xrightarrow{\delta} & H^{n+1}(N,M;\mathbb{R}) & \xrightarrow{q} & H^{n+1}(N;\mathbb{R}) & \longrightarrow \cdots \\
 & \downarrow D_N & & \downarrow D_M & & D_{N,M}\downarrow & & \downarrow D_N & \\
\cdots \longrightarrow & H_{n+1}(N,M;\mathbb{R}) & \xrightarrow{\partial} & H_n(M;\mathbb{R}) & \xrightarrow{i_*} & H_n(N;\mathbb{R}) & \xrightarrow{p} & H_n(M,N;\mathbb{R}) & \longrightarrow \cdots
\end{array}$$

The top row is the cohomology exact sequence of the pair (N,M), the bottom row the homology exact sequence of the same pair; and the vertical maps are duality isomorphisms: D_M is Poincaré duality, the remaining vertical maps Poincaré–Lefschetz duality (which we have not reviewed). Fix a complement K to L in $H^n(M;\mathbb{R})$. We shall show that $\dim K = \dim L$.

From exactness of the top row, we see that $L = \ker\delta$, so $K \cong \operatorname{im}\delta \cong \ker q$. But $\ker q \cong \ker p \cong \operatorname{im} i_*$, so $K \cong \operatorname{im} i_*$. Real cohomology is dual to real homology, and i^* is dual to i_*. Thus $\operatorname{im} i^*$ is the annihilator of $\ker i_*$, and $\dim \operatorname{im} i^* = \dim \operatorname{im} i_*$, i.e. $\dim L = \dim K$. □

1.1 Symmetric forms over $\mathbb{R}$

We concentrate now on the case where n is even, so Q_M is symmetric.

Definition 1.5 A *unimodular lattice* (Λ, σ) is a free abelian group Λ of finite rank, together with a non-degenerate, symmetric bilinear form $\sigma\colon \Lambda \times \Lambda \to \mathbb{Z}$.

Q_M is a unimodular lattice.

Recall that given a symmetric bilinear form σ on a real finite-dimensional real vector space V, there is an orthogonal decomposition

$$V = R \oplus V^+ \oplus V^-,$$

where $R = \{v \in V : \sigma(v,\cdot) = 0\}$ is the radical, and where σ is positive-definite on V^+ and negative-definite on V^-. The dimensions $\dim V^\pm$ are invariants of (V,σ), and together with that of R they are *complete* invariants.

We define the signature $\tau(\Lambda)$ of a unimodular lattice (Λ,σ) to be that of $\Lambda \otimes \mathbb{R}$, and the signature of M to be that of Q_M.

The fact that this $\tau(M)$ is an invariant of a closed oriented manifolds (of dimension divisible by 4) immediately gives the

Proposition 1.6 *A $4k$-dimensional closed oriented manifold M admits an orientation-reversing self-diffeomorphism only if its signature vanishes.*

Theorem 1.7 *(a) Let Y be a an oriented cobordism between 4-manifolds X_1 and X_2 (i.e., Y is a compact oriented 5-manifold with boundary ∂Y, together with an oriented diffeomorphism $\partial Y \cong -X_1 \amalg X_2$). Then $\tau_{X_1} = \tau_{X_2}$.*

(b) Conversely, if $\tau_{X_1} = \tau_{X_2}$, an oriented cobordism exists.

Proof (a) By the proposition above, the cup-product form of $-X_1 \amalg X_2$ admits a middle-dimensional isotropic subspace. It follows, as a matter of algebra, that $\tau(-X_1 \amalg X_2) = 0$. But the cup-product form $Q_{-X_1 \amalg X_2}$ is the orthogonal sum of $Q_{-X_1} = -Q_{X_1}$ and Q_{X_2}, so $\tau(-X_1 \amalg X_2) = \tau(X_2) - \tau(X_1)$.

(b) [Sketch.] It follows from Thom's cobordism theory that the group Ω_d of cobordism classes of closed oriented d-manifolds, under disjoint union, is isomorphic to the homotopy group $\pi_{d+k} M\,\mathrm{SO}(k)$ in the 'stable range' where k is reasonably large. Here $M\,\mathrm{SO}(k)$ is the Thom space of the universal vector bundle $E\,\mathrm{SO}(k) \to B\,\mathrm{SO}(k)$ over the classifying space for the Lie group $\mathrm{SO}(k)$. Note that the homology group $H_{d+k}(M\,\mathrm{SO}(k))$ is isomorphic (by the Thom isomorphism) to $H_d(B\,\mathrm{SO}(k))$; so there are Hurewicz maps $\Omega_d \to H_d B\,\mathrm{SO}(k)$, and in particular map $\Omega_4 \to H_4(B\,\mathrm{SO}(k)) \cong \mathbb{Z}$. Thom proves that $\Omega_d = 0$ for $d \leq 3$ and that $\Omega_4 \to \mathbb{Z}$ is an isomorphism. The signature homomorphism $\tau \colon \Omega_4 \to \mathbb{Z}$ is surjective, since $\tau(\mathbb{C}P^2) = 1$, and therefore an isomorphism. □

1.2 Characteristic vectors

Having examined Q_M over $\mathbb{R}$, we turn next to an aspect of its mod 2 arithmetic.

Definition 1.8 A *characteristic vector* c for a unimodular lattice is an element $c \in \Lambda$ such that $c \cdot x \equiv x \cdot x \mod 2$ for all $x \in \lambda$.

Lemma 1.9 *The characteristic vectors form a coset of 2Λ in Λ.*

Proof Let $\lambda = \Lambda \otimes_{\mathbb{Z}} (\mathbb{Z}/2)$. It is a $\mathbb{Z}/2$-vector space of dimension d, with a symmetric pairing $(\cdot,\cdot)$, still non-degenerate. The map $\lambda \to \lambda$ given by $z \mapsto (z,z)$ is $\mathbb{Z}/2$-linear, and so by non-degeneracy can be represented as $(z,z) = (\bar{c}, z)$ for a unique element $\bar{c} \in \lambda$. The characteristic vectors c are precisely the lifts of $\bar{c}$ to Λ. □

Definition 1.10 A unimodular lattice (Λ, σ) is called *even* if 0 is characteristic, i.e., if (x,x) is always even; otherwise the lattice is called *odd*. The property of being even or odd is called the *type* of the lattice.

Lemma 1.11 *For any two characteristic vectors c and c', one has $\sigma(c,c) \equiv \sigma(c',c')$ modulo 8.*

Proof Write $c' = c + 2x$. Then

$$\sigma(c',c') = \sigma(c,c) + 4(\sigma(c,x) + \sigma(x,x)),$$

and $\sigma(c,x) + \sigma(x,x)$ is even. □

Theorem 1.12 (Hasse–Minkowski) *A unimodular form σ on a lattice $\Lambda \cong \mathbb{Z}^r$, which is indefinite (i.e. neither positive- nor negative-definite) is determined, up to isomorphism, by its rank r, signature $\tau \in \mathbb{Z}$, and type $t \in \mathbb{Z}/2$.*

This is a deep and powerful result which we will not prove; see J.-P. Serre, *A course in arithmetic*. The key point is to find an isotropic vector, i.e. a vector $x \neq 0$ such that $\sigma(x,x) = 0$. It suffices to find an isotropic vector x in $\Lambda \otimes \mathbb{Q}$; and according Hasse–Minkowski's local-to-global principle for quadratic forms over $\mathbb{Q}$, for existence of such an isotropic vector it is necessary and sufficient that there are isotropic vectors in $\Lambda \otimes \mathbb{R}$ (to which indefiniteness is clearly the only obstruction) and in $\Lambda \otimes \mathbb{Q}_p$ for each prime p. Quadratic forms over the p-adics $\mathbb{Q}_p$ can be concretely understood, and it turns out that (when the rank is at least 5) there is a p-adic isotropic vector as soon as the form is indefinite (additional arguments are needed for low rank).

Let I_+ denote the unimodular lattice $\mathbb{Z}$ with form $(x,y) \mapsto xy$; let $I_- = -I_+$. Part of the statement of Hasse–Minkowski is that, if Λ is odd and indefinite, it is isomorphic to a direct sum

$$rI_+ \oplus sI_-$$

for suitable r and s. To prove this, one uses an isotropic vector to find an orthogonal direct sum decomposition $\Lambda = I_+ \oplus I_- \oplus \Lambda'$. Then $I_+ \oplus \Lambda'$ and $I_- \oplus \Lambda'$ have lower rank than Λ, and both are odd. One of them is indefinite, so one can proceed by induction on the rank.

The classification of odd indefinite unimodular forms has the following

Corollary 1.13 *In any unimodular lattice, any characteristic vector c has $\sigma(c,c) \equiv \tau$ mod 8. In particular, the signature of an even unimodular lattice is divisible by 8.*

Proof The form $rI_+ \oplus sI_-$ has characteristic vector $c = (1, \dots, 1)$, for which one has $c^2 = \tau$. Thus for any characteristic vector one has $c^2 \equiv \tau$ modulo 8. By the classification, the corollary holds for odd, indefinite unimodular forms. We can make any unimodular form odd and indefinite by adding I_+ or I_-, which has the effect of adding or subtracting 1 to the signature. If c is characteristic for Λ then $c \oplus 1$ is characteristic for $\Lambda \oplus I_\pm$, with $(c \oplus 1)^2 = c^2 \pm 1$, so we deduce the corollary for Λ. □

The basic example of an *even* unimodular form is the lattice $U = \mathbb{Z}^2$ with $(a,b)^2 = 2ab$. Its matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

To classify even indefinite unimodular forms one proceeds as follows. Suppose Λ_1 and Λ_2 are indefinite, unimodular and even, of the same rank and signature. One uses the existence of an isotropic vector to prove that $\Lambda_i \cong U \oplus \Lambda_i'$ for even unimodular lattices Λ_i'. From what has been proved about the odd case, one knows that $\Lambda_1' \oplus I_+ \oplus I_- \cong \Lambda_2' \oplus I_+ \oplus I_-$, and with some work one deduces that $\Lambda_1' \oplus U \cong \Lambda_2' \oplus U$, i.e., that $\Lambda_1 \cong \Lambda_2$.

1.3 The E_8 lattice

There is an important example of a positive-definite even unimodular form of rank 8. This is the form E_8 arising from the E_8 root system (or Dynkin diagram). Start with the lattice $\mathbb{Z}^8$ (standard inner product). Let $\Gamma \subset \mathbb{Z}^8$ be the sub-lattice formed by $x \in \mathbb{Z}^8$ with $x \cdot x$ even. Then E_8 is formed from Γ by adjoining the vector $\frac{1}{2}(e_1 + \cdots + e_8)$. Since this vector has length-squared 2, E_8 is even.

Exercise 1.14 (1) Let Λ be a lattice in $\mathbb{R}^n$ (with inner product inherited from $\mathbb{R}^n$) and $\Lambda' \subset \Lambda$ a sub-lattice of finite index $[\Lambda : \Lambda']$. Show that the determinants of the matrices representing these lattices are related by

$$\det \Lambda = [\Lambda : \Lambda'] \det \Lambda'.$$

(2) Show that $[\mathbb{Z}^8 : \Gamma] = 2$ and $[E_8 : \Gamma] = 2$.

(3) Deduce that $\det E_8 = 1$.

E_8 has basis $(v_1, \dots, v_8)$ where

$$v_i = e_{i+1} - e_i \quad (1 \le i \le 6), \quad v_7 = \tfrac{1}{2}(e_1 + e_8) - \tfrac{1}{2}(e_2 + \cdots + e_7), \quad v_8 = e_1 + e_2.$$

One has $v_i \cdot v_i = 2$; $v_1 \cdot v_2 = v_2 \cdot v_3 = \cdots = v_5 \cdot v_6 = -1$; $v_7 \cdot v_2 = -1$; $v_8 \cdot v_7 = 0$. All the other pairs are orthogonal. (One usually depicts this situation via the E_8 Dynkin graph.)

We typically prefer to use the negative-definite version $-E_8$. This has basis $(v_1, \dots, v_8)$ and matrix

$$-E_8 = \begin{bmatrix} -2 & 1 & & & & & 1 & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & & \\ & 1 & & & & & -2 & 1 \\ & & & & & & 1 & -2 \end{bmatrix}.$$

The direct sum

$$rU \oplus s(\pm E_8)$$

is even unimodular of rank $2r \pm 8s$ and signature $\pm 8s$. By Hasse–Minkowski and the fact that the signature of an even unimodular form is divisible by 8, we see that every indefinite even unimodular form takes this shape.

1.4 Topological examples

It is straightforward to write down an example of a $4k$-manifold with cup-product form H: one can simply take $S^{2k} \times S^{2k}$. In particular, in 4 dimensions we have $S^2 \times S^2$.

In 4 dimensions, it is also easy to come up with an example with cup-product for I_+: one can take $\mathbb{C}P^2$, with its orientation as a complex surface. One has $H^2(\mathbb{C}P^2) = \mathbb{Z}$, the generator ℓ being the Poincaré dual to any projective line $L \subset \mathbb{C}P^2$. Any two such lines, L and L', if distinct, intersect positively at a single point, so $\ell \cdot \ell = 1$. We can get I_- as intersection form by taking $-\mathbb{C}P^2$ (i.e. reversing orientation).

Next time, we shall use characteristic classes of the tangent bundle to prove the following

Proposition 1.15 *Let X be a smooth quartic complex surface in $\mathbb{C}P^3$. Then X has even intersection form of rank 22 and signature -16.*

Thus from Hasse–Minkowski, we deduce that X has intersection form

$$3U \oplus 2(-E_8).$$

It is not a simple task to write down an integral basis for $H_2(X)$, let alone to calculate the intersection form explicitly, so Hasse–Minkowski is a convenient shortcut.