4 The monopole TQFT

4.1 The monopole equations in 3 dimensions

There is a version of the Seiberg–Witten equations for a Riemannian 3-manifold \( Y \). A Spin\(^c\)-structure \( t \) on \( Y \) consists of a rank 2 hermitian vector bundle \( S \to Y \) (the spinor bundle) and an oriented isometry \( \rho : T^*Y \to su(S) \), where \( su(S) \) has the metric \( |a|^2 = -\text{tr} a^2 \). There’s a notion of a Clifford connection \( B \) in \( S \), and each of these has an associated Dirac operator

\[
D_B : \Gamma(S) \to \Gamma(S)
\]

The Clifford multiplication \( \rho \) extends naturally to a map

\[
\rho : \Lambda^Y \otimes \mathbb{C} \to \text{End}(S).
\]

The 3-dimensional Seiberg–Witten equations for a pair \((B, \psi)\) of Clifford connection and spinor \( \psi \in \Gamma(S) \) read

\[
(1) \quad D_B \psi = 0, \quad \frac{1}{2} \rho(F_B - i\eta) - (\psi^* \otimes \psi)_0 = 0.
\]

Here \( \eta \) is a chosen closed 2-form. We also impose a gauge-fixing condition

\[
d^*(B^0 - B_0^0) = 0.
\]

The linearized equations define an operator \( i\Omega^1 \oplus \Gamma(S) \to i\Omega^0 \oplus i\Omega^1 \oplus \Gamma(S) \) which cannot possibly be elliptic (the symbol maps between vector spaces of different dimension, so can’t be an isomorphism), but there’s an easy fix (including an extra scalar field \( f \) with \( df = 0 \)) that makes it elliptic.

4.2 Cylinders

On a cylinder \( Y \times \mathbb{R} \), with translation-invariant metric, the Seiberg–Witten equations arise as a gradient flow equation. Fix a Spin\(^c\)-structure \( t \) on \( Y \), i.e., a rank 2 hermitian
vector bundle $\mathbb{S} \to Y$ and an oriented isometry $\rho: T^*Y \to \mathfrak{su}(\mathbb{S})$. It extends naturally to a 4-dimensional Spin$^c$-structure on $Y \times \mathbb{R}$.

We consider the space $C(Y, t)$ of pairs $(B, \psi)$ of Clifford connection $B$ in $\mathbb{S}$ and $\psi \in \Gamma(\mathbb{S})$. There is a canonical bijection

$$C^\infty(\mathbb{R}, C(Y, t)) \leftrightarrow \{ (A, \phi) \in C(Y \times \mathbb{R}, s) : A \text{ in temporal gauge} \},$$

where temporal gauge means that $A$ has vanishing $dt$-component. In this temporal gauge, solutions $(A, \phi)$ to the monopole equations are the same thing as solutions to the equation

$$\frac{d}{dt}(B(t), \psi(t)) + \nabla L(B(t), \psi(t)) = 0$$

where $L$ is the Chern–Simons–Dirac functional,

$$L(B(t), \psi(t)) = -\frac{1}{8} \int_Y (iF_{B^t} + iF_{B_0^t} - \eta) \wedge (B^t - B_0^t) + \frac{1}{2} \int_Y \langle DB\psi, \psi \rangle d\text{vol}_g.$$ 

The critical points of this functional—stationary solutions to the 4D monopole equations—are the 3D monopole equations. (The failure of ellipticity is related to temporal gauge; stationary solutions could have a constant but non-zero term $c dt$ in the connection $B$.) $L$ is invariant under the identity component of the gauge group. For a general gauge transformation $u: Y \to U(1)$, defining a cohomology class $[u] = u^*(dt)$, one has

$$L(u \cdot (B, \psi)) - L(B, \psi) = 2\pi^2 \int_Y [u] \wedge c_1(\mathbb{S}) \in 2\pi\mathbb{Z}.$$ 

### 4.3 Monopole Floer homology

The gradient flow interpretation of the Seiberg–Witten equations is highly suggestive:

- If $X$ is a compact 4-manifold bounding $Y$, $\hat{X}$ its cylindrical completion, and $\eta_X$ a closed extension of $\eta$ to $X$, one should expect that monopoles on $\hat{X}$ (satisfying a ‘finite energy’ condition) will converge, in temporal gauge, to 3-dimensional $\eta$-monopoles on $Y$.
- One can expect to define an invariant $m(X, \eta_X) \in HM(Y, \eta)$, where $HM(Y, \eta)$, a ‘monopole homology group’, is the the ‘elliptic Morse–Novikov’ (or ‘Floer’) homology of the functional $L_\eta: \mathcal{B}(Y) \to S^1$.

The first clause is correct. The second clause is also correct if $[\eta] \in H^2(Y; \mathbb{R})$ is chosen so as to forbid, for Chern–Weil reasons, the existence of reducible monopoles $(B, \psi = 0)$.

In general, it is too naive because of the complicating effect of reducible monopoles, which have non-trivial stabilizer $U(1) \subset \mathfrak{g}_Y$. 

Roughly, the corrected version—say when $[\eta] = 0$—goes as follows.

Let $\tilde{B}(Y) = \mathcal{C}(Y)/\mathcal{G}_{Y, Y}$ be the quotient of the configuration space $\mathcal{C}(Y)$ by the free action of the based gauge group $\mathcal{G}_{Y, Y} = \{u \in \mathcal{G}_Y : u(y) = 1\}$. There is a residual action of $U(1)$ on $\tilde{B}(Y)$, and we wish to consider $U(1)$-equivariant Morse homology of $L$ on $P := \tilde{B}$.

Inside $P$, we have the locus $Q = P^{fix}$ of $U(1)$-fixed points. We have the homotopy quotient (or Borel construction)

$$P_{U(1)} = P \times_{U(1)} S^\infty,$$

and inside it the subspace

$$Q_{U(1)} = Q \times_{U(1)} S^\infty \cong Q \times \mathbb{C}P^\infty.$$

We shall be interested in the long exact sequence for homology of the pair,

$$\cdots \to H_*(Q_{U(1)}) \to H_*(P_{U(1)}) \to H_*(Q_{U(1)}, P_{U(1)}) \to \cdots.$$

More accurately, we shall be interested in a long exact sequence

$$\to \overline{HM}_*(Y) \to \widehat{HM}_*(Y) \to \widehat{HM}_*(Y) \to$$

of Morse–Floer homology groups for the functional $\mathcal{L}$ on $P$, constructed as ‘semi-infinite’ analogs of the homology groups of $Q_{U(1)}$, $P_{U(1)}$ and $(P_{U(1)}, Q_{U(1)})$.

These are the monopole Floer homology groups of $Y$. They are set up by Kronheimer–Mrowka in their book using a beautiful and unusual geometric model for $U(1)$-equivariant Morse–Floer homology of $L$.

**HM-bar.** The least interesting of the groups is $\overline{HM}_*(Y)$, which corresponds to $H_*(Q_{U(1)})$. It is constructed from solutions to a decoupled version of the monopole equations,

$$D_B \psi = \lambda \psi, \quad F_B = 0,$$

whose solutions are flat $U(1)$ connections $B$ and Dirac eigenspinors $\psi$. This group is determined entirely by $H^1(Y; \mathbb{Z})$ with the triple cup-product form $\Lambda^3 H^1(Y; \mathbb{Z}) \to \mathbb{Z}$.

### 4.4 The TQFT

There is a cobordism category $\text{COb}_{3+1}^{\text{con}}$ whose objects are closed, oriented, connected 3-manifolds. A morphism $Y_1 \to Y_2$ is a diffeomorphism-class of compact, oriented cobordisms from $Y_1$ to $Y_2$. Seiberg–Witten theory extends to a functor defined on
$C^\text{conn}_3^{cob}$. It assigns to each object $Y$ an exact triangle of abelian groups (actually, topological $\mathbb{Z}[[U]]$-modules)

$$\tilde{H}_* \to \overline{H}_* \to \hat{H}_*,$$

each graded by the set $\text{Spin}^c(Y)$.

A cobordism $X$ from $Y_1$ to $Y_2$ induces homomorphisms $HM(X): HM_*(Y_1) \to HM_*(Y_2)$ on each version of monopole Floer theory, respecting the exact triangles. Composition of cobordisms corresponds to composition of homomorphisms.

Each $\text{Spin}^c$-structure on $X$ gives rise to a map $HM(X, s)$ between appropriate summands of the $HM$-groups, and $HM(X)$ is the sum of all of these.

If one wants to incorporate disconnected $Y$ into the theory, a more elaborate algebraic construction will be needed, reflecting the structure of the cohomology of the ambient configuration space. I don’t know how to do this.

One might expect that the SW invariant of a closed $\text{Spin}^c$ 4-manifold $X$ would be obtained by applying one of these homomorphisms $HM(X^0, s)$ to the cobordism $X^0$ from $S^3$ to $S^3$ obtained by puncturing $X$ twice. Note, however, that the maps in the monopole Floer theory exist regardless of $b^+(X)$. And in fact the map $HM(X^0, s)$ is zero in all three theories.

The SW invariant of $X$ is extracted from the Floer theory by a ‘secondary cohomology operation’, secondary in the sense that it is only well-defined when $b^+(X) > 0$ (and is multi-valued if $b^+(X) = 1$). When $b^+(X) > 0$, there are no reducible solutions to the monopole equations. The map $\widehat{HM}(X, s)$, which counts only reducible solutions, is zero because its defining moduli spaces are empty. This emptiness facilitates the construction of a diagonal map

$$\tilde{HM}(X, s) \to \widehat{HM}(X, s)$$

factoring $\widehat{HM}(X, s)$. It is this diagonal map from which the Seiberg–Witten invariant can be extracted.