Broken pencils and four-manifold invariants

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Aim

This talk is about a project to construct and study a symplectic substitute for gauge theory in 2, 3 and 4 dimensions. The 3- and 4-manifolds are supposed to carry (possibly singular) fibrations by surfaces.

It is hoped that, by decomposing the fibrations into their simplest pieces, one can elucidate the structure of the theory.

Today I will tell you

(i) how Seiberg-Witten theory might be related to symplectic geometry on symmetric products of surfaces (this motivates the project);

(ii) about broken pencils on 4-manifolds;

(iii) part of the story of how to obtain Seiberg-Witten-like invariants for broken pencils via symplectic geometry.
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From gauge theory to symplectic geometry...

Witten: The moduli spaces of gauge-orbits of *Seiberg-Witten monopoles* over a smooth, oriented, closed 4-manifold X^4 lead to an invariants of X,

$$\mathrm{SW}_X \colon H^2(X;\mathbb{Z}) \to \mathbb{Z}.$$

Kronheimer-Mrowka: SW monopoles over a cylinder $Y^3 \times \mathbb{R} \rightsquigarrow$ *monopole Floer homology groups* $HM_\bullet(Y)$ (three variants). 4-manifolds bounding 3-manifolds give rise to maps between these groups. In this way we obtain a $(3+1)$-dimensional TQFT.

Can we go further and allow 3-manifolds to have boundary?

A 2-tier TQFT in (2+1+1) dimensions should assign to a surface Σ an additive (maybe A^∞) category $F(\Sigma)$, and to a cobordism from Σ to Σ' a functor $F(\Sigma) \to F(\Sigma')$.

Notionally: in a $(2+1+1)$-dimensional 2-tier Seiberg-Witten TQFT, we should assign to Σ^2 the *Donaldson-Fukaya category*

$$\mathcal{F}(\mathsf{S}^{k+g(\Sigma)}(\Sigma))$$

whose objects are Lagrangian submanifolds of the symmetric product $\mathsf{S}^{k+g(\Sigma)}(\Sigma)$. The morphisms are Lagrangian Floer homology groups $HF(L_1, L_2)$.

Here the symplectic form is a Kähler form in a certain cohomology class, and we get a different theory for each k.

Why this category?

Fix a surface Σ and a line bundle $L \to \Sigma$.

There's a 1-1 correspondence between $\mathbb{R}^2$-invariant SW monopoles in $L \to \Sigma^2 \times \mathbb{R}^2$, modulo gauge, and the symmetric product $\mathsf{S}^{\deg(L)}(\Sigma)$.

By considering SW monopoles over a 3-dimensional cobordism U, one obtains a Lagrangian submanifold $L_U \subset \mathsf{S}^n(\partial U)$ for a certain n.

There's an asymptotic relation between (non-translation-invariant) Seiberg-Witten monopoles on $\Sigma \times \mathbb{R}^2$ and pseudo-holomorphic curves in Σ (these are involved in the constructions of $HF_*(L_1, L_2)$).

...And back again?

Can we rebuild Seiberg-Witten theory symplectically, starting from symmetric products and their Lagrangians?

Conjecturally, yes:

Ozsváth-Szabó: use Heegaard diagrams on 3-manifolds. A Heegaard diagram gives a pair of Lagrangian g-tori $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$ in $\mathrm{S}^g(\Sigma)$, where Σ is the Heegaard surface, with Floer homology $HF^\infty(Y) := HF_*(\mathbb{T}_\alpha, \mathbb{T}_\beta)$.

P.: use S^1-valued Morse functions without extrema on 3-manifolds and *broken fibrations* on 4-manifolds. A critical point of the Morse function gives rise to a Lagrangian correspondence between two symmetric products.

In a precise sense, the second method contains the first.

Why would we want to do this?

The Ozsváth-Szabó theory has had great success in making Floer homology for 3-manifolds computable and 'quasi-combinatorial'. It may soon be possible to reprove Donaldson's theorems about 4-manifolds without using analysis.

My motivation was not to make SW 'geometric' (in the vein of Taubes' programme in near-symplectic geometry and the Donaldson-Smith invariants for Lefschetz pencils).

A current aim: Make SW algebraic by elucidating its TQFT structure in presence of singular fibrations.

Becoming plausible: compute $HM_\bullet(Y)$ when Y is fibred—express it as the mapping torus of a composite of Dehn twists on a surface.

Conceivable: compare Seiberg-Witten and instanton TQFTs when the manifolds carry singular fibrations.

Pencils and fibrations

A *pencil* on a smooth oriented 4-manifold X is given by a finite subset $B \subset X$ and a smooth map $\pi \colon X \backslash B \to \mathbb{CP}^1$ such that, near each $b \in B$, there are local complex coordinates (z_1, z_2) in which the map is given by

$$(z_1, z_2) \mapsto (z_1 : z_2).$$

Moreover, the fibres $\pi^{-1}(x)$ must have compact closure in X.

A *fibration* is a smooth, proper surjective map.

We can convert a pencil to a fibration by blowing up along B. Replace X by its blow-up $\widehat{X} = X \# k\overline{\mathbb{CP}}^2$, $k = |B|$. The blow-down map composed with π extends smoothly to give a fibration $\widehat{X} \to S^2$.

Special kinds of pencils and fibrations

Lefschetz pencils/fibrations. The extra requirement here is that, for any critical point c of π, there are oriented complex coordinates (z_1, z_2) near c, and a complex coordinate near $\pi(c)$, in which the map is given by

$$(z_1, z_2) \mapsto z_1 z_2.$$

So the critical fibres have isolated nodal singularities.

Donaldson (mid 1990s) proved that any closed symplectic 4-manifold (X, ω) *admits a Lefschetz pencil such that* ω *is positive on the fibres.*

Broken pencils/fibrations. Here the critical set crit(π) is required to be a union of isolated points and circles, mapping injectively to S^2. Near an isolated critical point c, the model is as for a Lefschetz pencil:

$$\mathbb{C}^2 \ni (z_1, z_2) \mapsto z_1 z_2 \in \mathbb{C}.$$

Near a non-isolated critical point, the model is the map

$$\mathbb{R}\times\mathbb{R}^3 \ni (t, x_1, x_2, x_3) \mapsto (t, \pm(x_1^2+x_2^2-x_3^2)) \in \mathbb{R}^2.$$

Donaldson, Auroux and Katzarkov (2004) proved that every closed, oriented 4-manifold whose intersection form is not negative-definite admits a broken pencil.

Along a path in S^2 which transversely crosses a critical circle, the topology of the fibre changes (going in one direction, the genus drops by one).

Elementary cobordisms between surfaces

Fix a 'level' k. Assign to each closed Riemann surface Σ_g of genus g the complex manifold $\mathsf{S}^{g+k}(\Sigma_g)$. Choose a Kähler form ω (from certain preferred cohomology classes).

Consider an elementary cobordism W from Σ_{g+1} to Σ_g. There's a Morse function $f\colon W \to \mathbb{R}$ with just one critical point, of index 1. We can essentially reconstruct (W, f) by giving Σ and the 'attaching circle' $\Gamma \subset \Sigma_{g+1}$.

In particular, Σ_g is the result of surgery on Σ_{g+1} along Γ.

Lagrangian correspondences

Take a Riemann surface Σ_{g+1} with an embedded circle Γ; construct Σ_g from them. Construct $\mathsf{S}^{k+g+1}(\Sigma_{g+1})$ and $\mathsf{S}^{k+g}(\Sigma_g)$ as Kähler manifolds.

Theorem. There is a canonical Hamiltonian isotopy-class of Lagrangian submanifolds

$$V_\Gamma \subset \mathsf{S}^{k+g+1}(\Sigma_{g+1}) \times -\mathsf{S}^{k+g}(\Sigma_g),$$

with the following two properties:

(i) The projection $p_1\colon V_\Gamma \to \mathsf{S}^{k+g+1}(\Sigma_{g+1})$ is a codimension 1 embedding.

(ii) The projection $p_2\colon V_\Gamma \to \mathsf{S}^{k+g}(\Sigma_g)$ is an S^1-bundle.

We can think of V_Γ as being attached to the 3-cobordism W. However, the proof does NOT use W. It uses the symplectic geometry of an algebro-geometric degeneration of $\mathsf{S}^{k+g+1}(\Sigma_{g+1})$: a family $\mathcal{E} \to \Delta$ over the unit disc whose smooth fibre $\mathcal{E}$ is $\mathsf{S}^{k+g+1}(\Sigma_{g-1})$ and whose central fibre has singular locus $\mathsf{S}^{k+g}(\Sigma_g)$. V_Γ is its *vanishing cycle*.

What does $p_1(V_\Gamma) \subset \mathsf{S}^{k+g+1}(\Sigma_{g+1})$ look like? Some points of it consist of $k+g$ points far from Γ plus one point of Γ. But when there are two or more points near Γ it becomes harder to be explicit.

In principle, V_Γ should define a functor

$$\mathcal{F}(\mathsf{S}^{k+g}(\Sigma_g)) \to \mathcal{F}(\mathsf{S}^{k+g+1}(\Sigma_{g+1})) :$$

take a Lagrangian submanifold $L \subset \mathsf{S}^{k+g}(\Sigma_g)$, and send it to $V_\Gamma(L) = p_1(p_2^{-1}(L))$.

Composing Lagrangian correspondences

Choose a g-tuple of disjoint circles $\Gamma_1, \ldots, \Gamma_g \subset \Sigma_g$, linearly independent in homology. We get a sequence of surfaces Σ_g, $\Sigma_{g-1}, \ldots \Sigma_0 \cong S^2$ and cobordisms between them. The composite cobordism is a genus g *handlebody* U.

At level 0, we also get correspondences

$$\begin{aligned} V_{\Gamma_g} &\subset \mathrm{S}^g(\Sigma_g) \times \mathrm{S}^{g-1}(\Sigma_{g-1}), \\ V_{\Gamma_{g-1}} &\subset \mathrm{S}^{g-1}(\Sigma_{g-1}) \times \mathrm{S}^{g-2}(\Sigma_{g-2}), \\ &\vdots \\ V_{\Gamma_1} &\subset \mathrm{S}^1(\Sigma_1) \times \mathrm{S}^0(\Sigma_0) = \Sigma_1 \times \{pt.\}. \end{aligned}$$

The composed correspondence $V_{\Gamma_1} \circ \cdots \circ V_{\Gamma_g} \subset \mathrm{S}^g(\Sigma) \times \{pt.\}$ is Lagrangian isotopic to the Heegaard torus

$$\Gamma_g \times \cdots \times \Gamma_1 \subset \mathrm{S}^g(\Sigma_g)$$

which appears in Ozsváth-Szabó's Heegaard Floer homology theory.

Lagrangian matching invariants

The Lagrangian correspondences I have described can be used to construct a TQFT-type theory. This assigns a symplectic Floer homology group to a 3-manifolds equipped with S^1-valued Morse functions with critical points of index 1 and 2. One gets a homomorphism between the Floer homology group from a broken fibration.

From broken fibrations on closed 4-manifolds one gets numerical invariants which are conjecturally the Seiberg-Witten invariants.

The topological invariance properties of this theory are not yet established.

Lagrangian matching invariants

For simplicity, consider a broken fibration $X \to S^2$ with just one critical circle (mapping to the equator in S^2) and suppose the fibres have genus $g+1$ over the northern hemisphere N, genus g over the southern hemisphere S.

Over the closed region N' north of the Tropic of Cancer, replace each fibre Σ by $\mathrm{S}^{n+g+1}(\Sigma)$. Over the closed region S' south of the Tropic of Capricorn, replace each fibre Σ' by $\mathrm{S}^{n+g}(\Sigma')$. (Isolated singular fibres—technical substitute due to Donaldson-Smith.)

Point x on the equator $\rightsquigarrow$ x_N, x_S: points of the same longitude lying on the tropics. Use the theorem to attach to x a Lagrangian correspondence $V_x \subset (\mathcal{E}_N)_{x_N} \times -(\mathcal{E}_S)_{x_S}$.

We now have fibrations $\mathcal{E}_N \to N'$ and $\mathcal{E}_N \to S'$, each of which we can make globally symplectic.

We can arrange that the V_x's form a Lagrangian submanifold V of the fibre product of $\mathcal{E}_N$ and $\iota^*\mathcal{E}_S$ (where ι identifies N' with S').

Choose almost complex structures on $\mathcal{E}_N$ and $\mathcal{E}_S$, making the projections to N' and S' pseudo-holomorphic (i.e. the derivative is $\mathbb{C}$-linear).

Consider pairs (u_N, u_S) of sections of $\mathcal{E}_N \to N'$ and $\mathcal{E}_S \to S'$ such that

(i) u_N and u_S are pseudo-holomorphic; and

(ii) $(u_N(x_N), u_S(x_S)) \in V_x$ for each point x of the equator.

Pairs (u_N, u_S) form a finite-dimensional moduli space which has good compactness and transversality properties.

Its local dimension can be computed in terms of topological data on X by a formula familiar from Seiberg-Witten theory. By 'counting' points in 0-dimensional components of the moduli space, one can derive an invariant of the fibration, the *Lagrangian matching invariant*. It has the same format as (part of) the Seiberg-Witten invariant, and I conjecture that equality holds.

This conjecture is supported by various computations, by formal properties, and by heuristic relations both to Seiberg-Witten theory and to Taubes' conjectural interpretation of it in near-symplectic geometry.