Eigenspinors and the Chern-Simons-Dirac functional

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Monopole Floer Homology reading group

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Monopole Floer homology assigns to a Spin^c-3-manifold an exact triangle of $\mathbb{Z}[[U]]$-modules

$$\cdots \to \widehat{HM}_\bullet \to \overline{HM}_\bullet \to \widecheck{HM}_\bullet \to \cdots .$$

They are homology modules for chain complexes generated by critical points of the blown up Chern-Simons-Dirac functional. Each involves two out of the three types of critical points: irreducible, boundary-stable and boundary-unstable.

Why so complicated? If one took only the irreducible ones, say, the homology group would not be metric-independent.

In today's lecture we'll begin to analyse the boundary critical points, and take a first look at the situation on S^3. In that case, the exact triangle can be identified with the short exact sequence

$$0 \to \mathbb{Z}[[U]] \to \mathbb{Z}[U^{-1}, U]] \to \mathbb{Z}[U^{-1}, U]/\mathbb{Z}[U] \to 0.$$

Blowing up S^1-manifolds

Let P be a (compact) Riemannian manifold on which S^1 acts by isometries. Assume that the action is free away from the fixed-point set Q. By linearising the S^1 action, the normal bundle $N = N_{Q/P}$ becomes a complex vector bundle.

We wish to replace P by a free S^1-space. One possibility is the Borel construction $P \times_{S^1} S^\infty$, but a more economical alternative in this 'semi-free' case is to form the *real oriented blow-up* P^σ of P along Q, which is the smooth manifold-with-boundary

$$P^\sigma = (P \setminus Q) \cup_{p^\sigma} S(N) \times [0, \epsilon).$$

Here $p^\sigma \colon S(N) \times (0, \epsilon) \to P \setminus Q$ is the tubular neighbourhood embedding induced by the metric. There is a blow-down map $P^\sigma \to P$, which gives a diffeomorphism $\mathrm{int}(P^\sigma) \to P \setminus Q$. The map p^σ interwines the linear S^1 action on $S(N)$ with the action on P, hence P^σ is a free S^1-space.

Chern-Simons-Dirac

As an example, take the Hilbert manifold $\tilde{\mathcal{B}} = \mathcal{C}/\mathcal{G}_0$ associated with a closed 3-manifold Y with Spin^c-structure $\mathfrak{s}$. Here $\mathcal{C}$ is the configuration space of pairs (B, Φ) of Spin^c-connection and spinor of Sobolev class $L^2_{3/2}$, with the L^2 metric. The gauge group $\mathcal{G}$ has a subgroup $\mathcal{G}_0$ of gauge transformations u with $u(y) = 1$ at some basepoint $y \in Y$, and $\mathcal{G}_0$ acts freely. This leaves a residual S^1-action on $\mathcal{B}(Y)$ by constant gauge transformations.

The blow-up $\tilde{\mathcal{B}}^\sigma$ of $\tilde{\mathcal{B}}$ along the fixed locus (of reducible configurations $(B, 0)$) is a Hilbert manifold with boundary. So is its space of S^1-orbits $\mathcal{B}^\sigma$.

We next want to bring the Chern-Simons Dirac functional $\mathcal{L}$ into the picture. This function on $\mathcal{C}$ is $\mathcal{G}$-invariant, modulo $2\pi\,\mathrm{div}(c_1(\mathfrak{s}))$, and hence determines an S^1-invariant, circle-valued function on $\tilde{\mathcal{B}}$.

Morse theory on P^σ...

From an S^1-invariant function $\tilde{f} \in C^\infty(P)$.

$\tilde{V} :=$ grad$(\tilde{f})$ lifts to a vector field $\tilde{V}^\sigma$ on int(P^σ). Fact: $\tilde{V}^\sigma$ extends smoothly to a vector field on P^σ, tangent to ∂P^σ.

We study the critical points and gradient flows of V^σ, the vector field on P^σ/S^1 induced by $\tilde{V}^\sigma$. In a geodesic tubular neighbourhood of Q, we can write

$$\tilde{f}(q, \phi) = (\tilde{f}|_Q)(q) + \frac{1}{2}\langle L_q\phi, \phi\rangle + o(|\phi|^2).$$

Here the $\mathbb{C}$-linear, $\mathbb{R}$-symmetric map $L \in \mathrm{End}(N)$ is the Hessian of $\tilde{f}$. The behaviour of the stationary points of $\tilde{V}^\sigma$ on ∂P^σ can be understood in terms of Morse theory of $\tilde{f}|_Q$, and of the functions

$$N_q \to \mathbb{R}, \quad p \mapsto \frac{1}{2}\langle L_q p, p\rangle.$$

Consider a hermitian operator L on $\mathbb{C}^n$, and the function $l\colon p \mapsto \frac{1}{2}\langle Lp, p\rangle$. We have grad$(l)(p) = Lp$, so if L is non-degenerate its only critical point is the origin.

In polar coordinates $(\phi, s) \in S^{2n-1} \times (0, \infty)$, the gradient is

$$(\phi, s) \mapsto (L\phi - \langle \phi, L\phi\rangle \phi, \langle \phi, L\phi\rangle s).$$

Thus the lift of the gradient flow to the blow-up $S^{2n-1} \times [0, \infty)$ has stationary points $(\phi, 0)$, where ϕ is a unit eigenvector for L.

The stationary points of the induced field on the S^1-quotient $\mathbb{CP}^{n-1} \times [0, \infty)$ are in bijection with the eigenvalues of L, counted with multiplicities.

To analyse critical points of $\tilde{V}^\sigma$, represent points of P^σ in the form (q, ϕ, s) where $q \in Q$, $\phi \in S(N_q)$, $s \geq 0$. We have

$$T_{(q,\phi,s)} P^\sigma = T_q Q \oplus W \oplus \mathbb{R},$$

where $W = \phi^\perp$ is the tangent space to $S(N_q)$. We then have

$$\tilde{V}^\sigma(q, \phi, 0) = (\mathrm{grad}(\tilde{f}|_Q), L_q\phi - \langle \phi, L_q\phi \rangle \phi, 0).$$

So boundary critical points of $\tilde{V}^\sigma$ are precisely the points $(q, \phi, 0)$ where ϕ is a unit eigenvector of L_q.

At a stationary point $(q, \phi, 0)$ with $L_q\phi = \lambda\phi$, the Hessian is of shape

$$\nabla \tilde{V}^\sigma = \begin{bmatrix} \mathrm{Hess}(\tilde{f}|_Q)(q) & 0 & 0 \\ * & L_q - \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

This is still valid on P^σ / S^1: just take W to be the *complex* orthogonal complement $[\phi]^\perp$.

The Hessian computation shows that

- The stationary points of V^σ on $\partial P^\sigma/S^1$ are non-degenerate when (i) $\tilde{f}|_Q$ is Morse; and for each of its critical points q, the operator L_q has (ii) simple spectrum and (iii) no kernel.

- The stationary point $(q, [\phi], 0)$ is then boundary-stable—i.e. the outward normal to ∂P^σ is tangent to its stable manifold—iff it corresponds to a *positive* eigenvalue of L_q.

- Choose a basis of unit eigenvectors ϕ_i for L_q with eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n.$$

Then $\mathrm{ind}(q, [\phi_i], 0)$ is given by

$$\begin{cases} \mathrm{ind}_Q(q) + 2(i-1), & \lambda_i > 0, \\ \mathrm{ind}_Q(q) + 2(i-1) + 1, & \lambda_i < 0. \end{cases}$$

CSD again

The fixed-point set of S^1 acting on $\tilde{\mathcal{B}}$ is the locus of reducible configurations $(B, 0)$. We have $\mathcal{L}(B, 0) = \text{CS}(B)$ (Chern-Simons functional). Thus the critical points of $\mathcal{L}|_Q$ are gauge-orbits of *flat* S^1*-connections*, and $\mathcal{L}|_Q$ is Morse iff $b_1(Y) = 0$. The normal operator L_B at a flat connection is the Dirac operator D_B.

Thus the vector field $\mathcal{V}^\sigma$ on $\mathcal{B}^\sigma = \tilde{\mathcal{B}}^\sigma / S^1$ has as its boundary stationary points pairs $(B, [\Phi])$ where B^t is flat and $D_B \Phi = \lambda \phi$ for some $\lambda \in \mathbb{R}$.

Moreover,

These stationary points are non-degenerate when (i) $b_1(Y) = 0$, (ii) D_B has no kernel, and (iii) D_B has simple eigenvalues. In this case, they are boundary stable or unstable according to the sign of the eigenvalue.

For each flat B, there is a countable infinity of boundary-stable stationary points, likewise boundary-unstable. The non-degeneracy conditions will often not be met, and for this reason one considers perturbations of $\mathcal{L}$.

The 3-sphere

On S^3 with its standard metric g and unique Spin^c-structure, there are no irreducible SW monopoles as g has positive scalar curvature.

There's only one flat connection B_0, up to gauge. Its Dirac operator D_{B_0} has kernel zero (Lichnérowicz formula) but it does not have simple eigenvalues. It can be perturbed to a self-adjoint operator L which has simple, doubly-infinite spectrum and no kernel.

The stationary points $(B_0, [\phi_i])$ do not have well-defined Morse indices, but any pair of them do have a *relative* index. The relative index is, e.g., $2(i-j)$ when ϕ_i and ϕ_j are boundary-stable (cf. finite dimensional indices).

These observations lead to a simple computation of the three monopole Floer homology groups of S^3. Each of the three chain complexes is concentrated in degrees of the same parity, and so has trivial differentials.