Flat connections, Dirac operators and the triple cup-product form

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Introduction

This talk is about topology related to $\overline{HM}_*(Y, \mathfrak{s})$, the simplest of the three variants of monopole Floer homology, which is non-zero only if $c_1(\mathfrak{s})$ is a torsion class. $\overline{HM}_*$ is an elliptic Morse homology theory for monopoles in the boundary of the blown-up configuration space, i.e. pairs (B, ϕ) modulo gauge, where

- B is a flat Spin^c-connection and
- ϕ is a D_B-eigenspinor with $\|\phi\|_{L^2} = 1$.

Theorem (Kronheimer–Mrowka). When $c_1(\mathfrak{s})$ is torsion, $\overline{HM}_*(Y, \mathfrak{s})$ is determined up to isomorphism by the free abelian group $H^1(Y; \mathbb{Z})$ together with its natural alternating cubic form $\xi_3 \in \Lambda^3 H^1(Y; \mathbb{Z})^*$, the triple cup-product.

When $\xi_3 = 0$, e.g. when $b_1(Y) < 3$,

$$\overline{HM}_*(Y, \mathfrak{s}) \cong \Lambda^* H^1(Y; \mathbb{Z}) \otimes \mathbb{Z}[T, T^{-1}],$$

where T has degree -2.

In general, there is a decreasing filtration of $\overline{HM}_*(Y, \mathfrak{s})$ whose associated graded group is isomorphic to

$$\mathsf{H}_*(\Lambda^* H^1(Y; \mathbb{Z}), \beta) \otimes \mathbb{Z}[T, T^{-1}],$$

for a differential $\beta \colon \Lambda^i H^1(Y; \mathbb{Z}) \to \Lambda^{i-3} H^1(Y; \mathbb{Z})$ defined explicitly using ξ_3.

Non-vanishing

Lemma. Let $\beta\colon \Lambda V \to \Lambda V$ be a complex: $\beta^2 = 0$. Suppose β lowers degree by 3. Then $\mathrm{H}_*(\Lambda V, \beta)$ cannot be zero.

For there are three subcomplexes (fix the degree mod 3) and for these to be acyclic their Euler characteristics $\chi_{n,k} := \sum_i (-1)^i \binom{n}{3i+k}$ must vanish (here $n = \dim(V)$ and $k = 0, 1, 2$). But then $(1+e^{2\pi i/3})^n = \chi_{n,0} + \chi_{n,1} e^{2\pi i/3} + \chi_{n,2} e^{4\pi i/3}$ equals zero, which is absurd.

We deduce the

Theorem. $\overline{HM}_i(Y, \mathfrak{s})$ has positive rank in infinitely many positive and negative degrees i.

Since $\widehat{HM}_*(Y, \mathfrak{s})$ is supported in degrees bounded above, and $\check{HM}_*(Y, \mathfrak{s})$ in degrees bounded below, the exact triangle implies that these groups also have infinite rank.

This theorem is lurking behind Taubes' proof of the Weinstein conjecture.

Topology of spaces of operators

Let H be an infinite-dimensional separable Hilbert space. We consider subspaces of $\mathcal{B}(H)$ with its operator-norm topology.

Kuiper: the unitary group $U(H)$ is contractible.

Its subgroup $U = U(\infty) = \bigcup_n U(n)$ is *not* contractible. It is rationally equivalent to the product of odd spheres, $S^1 \times S^3 \times S^5 \times \ldots$.

Atiyah: The Fredholm operators form a classifying space Fred(H) for K^0 (the index bundle is the universal stable bundle). Hence

$$\text{Fred}(H) \simeq \mathbb{Z} \times BU.$$

Anon.: There's a homotopy equivalence $\Omega BU \simeq U$ (monodromy of the universal bundle pulled back to S^1), hence U classifies K^{-1}.

Bott: there is a homotopy equivalence

$$\Omega U \simeq \mathbb{Z} \times BU.$$

Self-adjoint Fredholm operators

Atiyah-Singer: The space $S(H)$ of self-adjoint Fredholm operators has three components. Two of these,

$$S_+(H) = \{L \in S(H) : L \text{ essentially positive}\},$$
$$S_-(H) = \{L \in S(H) : L \text{ essentially negative}\},$$

are contractible. The third component $S_*(H)$ is more interesting: the map

$$S_*(H) \to \Omega\,\mathrm{Fred}_0(H), \quad L \mapsto \alpha_L,$$

where α_L is the path from $+I$ to $-I$ given by $[0,1] \ni t \mapsto \cos(\pi t)I + \sin(\pi t)L$, is a homotopy equivalence.

Hence $S_*(H) \simeq \Omega BU \simeq U$ classifies $K^{-1} \cong K^1$.

The difficulty in applying Atiyah-Singer's result to gauge theory is that Dirac operators are unbounded.

Kronheimer-Mrowka: Let $H_1 \subset H$ be a dense subspace such that the inclusion is a compact operator. Let $S_*(H : H_1)$ be the space of index 0 Fredholm operators $L\colon H_1 \to H$ with $\langle Lx, y\rangle = \langle x, Ly\rangle$ whose spectrum is doubly infinite and which satisfy a certain technical condition. Then, under an additional hypothesis on H_1, $S_*(H : H_1) \simeq U$.

The Picard torus

Let $(Y^3, \mathfrak{s})$ be a closed, Spin^c 3-manifold. Set

$$\tilde{\mathbb{T}} = \{\text{Spin}^c\text{-connections } B \text{ in } S : F_{B^t} = 0\}.$$

This is non-empty iff $c_1(\mathfrak{s})$ is a torsion class. Let $B_0 \in \tilde{\mathbb{T}}$. Then $B \in \tilde{\mathbb{T}}$ iff $d(B^t - B_0^t) = 0$. We can find a gauge transformation u in the identity component $\mathcal{G}_0 \subset \mathcal{G}$ so that $d^*(uB^t - B_0^t) = 0$. Hence $\tilde{\mathbb{T}}/\mathcal{G}_0 = \mathcal{H}^1_g(Y) = H^1(Y; \mathbb{R})$,

$$\mathbb{T} := \tilde{\mathbb{T}}/\mathcal{G} \cong \frac{H^1(Y; \mathbb{R})}{H^1(Y; \mathbb{Z})}.$$

Family of Dirac operators

The torus $\mathbb{T}$ carries a family of Dirac operators $D_B \in S_*(H : H_1)$, where $H = L^2(S)$ and $H_1 = L^2_1(S)$.

This family is classified by a map $\mathbb{T} \to U$, or equivalently by a class $\zeta \in K^{-1}(\mathbb{T})$. We will describe ζ.

Since $H^*(\mathbb{T};\mathbb{Z})$ is torsion-free, ζ is determined by its Chern character

$$\mathrm{ch}(\zeta) \in H^{\mathrm{odd}}(\mathbb{T};\mathbb{Q}).$$

Applying the family index theorem

Theorem. *The Chern character* ch(ζ) *is equal to the class*

$$\xi_3 \in H^3(\mathbb{T};\mathbb{Z}) = \Lambda^3 H^1(\mathbb{T};\mathbb{Z}) = \Lambda^3 H^1(Y;\mathbb{Z})^*$$

given by the triple cup-product

$$(a,b,c) \mapsto \int_Y a \cup b \cup c.$$

This is proved using a version of the Atiyah-Singer index theorem which applies to families of self-adjoint Fredholm operators. Let $\mathcal{S} \to \mathbb{T} \times Y$ be the bundle which carries the universal family of flat connections. The index theorem says that

$$\mathrm{ch}(\zeta) = \int_Y \mathrm{ch}(\mathcal{S})\widehat{A}(Y) = \int_Y \mathrm{ch}(\mathcal{S}).$$

There is a (Poincaré) line bundle $\mathcal{P} \to \mathbb{T} \times Y$ such that $\mathcal{S} = \mathcal{P} \otimes p_2^* S$. We have $c_i(\mathcal{S}) = c_i(\mathcal{P})$, hence

$$\mathrm{ch}(\zeta) = \int_Y \mathrm{ch}(\mathcal{P}) = \sum_{n \geq 0} \frac{1}{n!} \int_Y c_1(\mathcal{P})^n.$$

We may assume $\mathcal{P}|_{\mathbb{T}\times\{y\}}$ is trivial; then $c_1(\mathcal{P})$ is the tautological class

$$\begin{aligned}\sum \alpha_i^* \otimes \alpha_i &\in H^1(Y;\mathbb{Z})^* \otimes H^1(Y;\mathbb{Z}) \\ &= H^1(\mathbb{T};\mathbb{Z}) \otimes H^1(Y;\mathbb{Z}) \\ &\subset H^2(\mathbb{T}\times Y;\mathbb{Z}).\end{aligned}$$

Here (α_i) is a basis for $H^1(Y;\mathbb{Z})$, (α_i^*) the dual basis. So

$$\begin{aligned}\int_Y \mathrm{ch}(\mathcal{P}) &= \frac{1}{6}\int_Y c_1(\mathcal{P})^3 \\ &= \sum_{i_1<i_2<i_3} \left(\int_Y \alpha_{i_1} \cup \alpha_{i_2} \cup \alpha_{i_3}\right) \alpha_{i_1}^* \cup \alpha_{i_2}^* \cup \alpha_{i_3}^* \\ &= \xi_3.\end{aligned}$$

Corollary. The classifying map $\mathbb{T} \to U(\infty)$ for the family of Dirac operators over $\mathbb{T}$ factors through $SU(2)$.

Sketch proof. ξ_3 is the pullback $v^*\eta$ of the orientation class $\eta \in H^3(SU(2);\mathbb{Z})$ by a map $v\colon \mathbb{T} \to SU(2)$. But $\eta = \mathrm{ch}(e)$ where $e \in K^{-1}(SU(2))$ corresponds to the restriction of the universal family over $U(\infty)$. Hence $\mathrm{ch}(\zeta) = \mathrm{ch}(v^*e)$, so $\zeta = v^*e$.

Monopole Floer homology and the Dirac family

The group $\overline{HM}_*(Y, \mathfrak{s})$ is built from critical points and trajectories of a vector field on the boundary of the blown-up configuration space modulo gauge. Before perturbation, the critical points are gauge-orbits $[B, \phi]$, where $[B] \in \mathbb{T}$, $\|\phi\|_{L^2} = 1$, and $D_B\phi = \lambda\phi$ for some $\lambda \in \mathbb{R}$.

We have just seen that the family of Dirac operators $D_\bullet$ over $\mathbb{T}$ is parametrised by the map $v \colon \mathbb{T} \to SU(2)$ which pulls back the orientation class to the triple cup-product class ξ_3.

Fact: $\overline{HM}_*(Y, \mathfrak{s})$ can be understood as a Morse homology group for $\mathbb{T}$ 'coupled' to ξ_3.

K-M give a general construction (over $\mathbb{Z}$) of Morse homology coupled to a K^{-1}-class. This is a form of twisted cohomology, analogous to K-theory twisted by a class in H^3. It seems likely that the K^{-1} class only enters through its Chern classes in $H^{odd}(-; \mathbb{Z})$.

Coupled Morse homology

Let Q be a compact, oriented manifold. Let $\xi \in [X, U(\infty)]$. We build the coupled Morse homology $H_*^{\xi}(Q)$.

Choose a metric g, a Morse-Smale function f and a smooth representative of ξ. The latter determines a map $Q \to S_*(H : H_1)$, $q \mapsto L_q$. For generic perturbations of this map, the operator L_q is non-degenerate for each $q \in$ crit(f). That is, L_q has simple eigenvalues and ker$(L_q) = 0$.

- Define a 'critical point' to be a pair (q, λ) where $q \in$ crit(f), $\lambda \in$ spec(L_q).

- Define a 'gradient trajectory' to be a map $(q, \phi) \colon \mathbb{R} \to Q \times H_1$ such that

$$\dot{q}(t) + \text{grad}(f)(q(t)) = 0,$$
$$q^*(\nabla^{LC})\phi + L_{q(t)}\phi \, dt = 0.$$

Say $L_{q_0}\psi_0 = \lambda_0\psi_0$, $L_{q_1}\psi_1 = \lambda_1\psi_1$. Define the moduli space $M(q_0, \lambda_0; q_1, \lambda_1)$ to be the space of $\mathbb{C}^*$-orbits of gradient trajectories (γ, ϕ), where γ connects q_0 to q_1, and

$$\begin{aligned} \phi(t) &\sim c_0 e^{-\lambda_0 t}\psi_0 \quad \text{as } t \to -\infty, \\ \phi(t) &\sim c_1 e^{-\lambda_1 t}\psi_1 \quad \text{as } t \to +\infty. \end{aligned}$$

These spaces have Fredholm deformation theory. The virtual dimension of $M(q_0, \lambda_0; q_1, \lambda_1)$ is

$$\operatorname{ind}(q_0) - \operatorname{ind}(q_1) + 2m,$$

where m is related to the spectral flow of $t \mapsto L_{\gamma(t)}$.

$H_*^\xi(Q)$ is defined as the homology of the complex freely generated by critical points, where the matrix entries of the differential count points in 0-dimensional moduli spaces $M(q_0, \lambda_0; q_1, \lambda_1)/\mathbb{R}$.

Kronheimer and Mrowka prove the following:

(a) When ξ factors through $SU(2)$, $H_*^{\xi}(Q)$ is isomorphic to the homology of a chain complex

$$C_*(Q,f) \otimes \mathbb{Z}[T,T^{-1}], \quad x \mapsto \partial x + T\tilde{\xi} \cap x$$

where $(C_*(Q,f),\partial)$ is the usual Morse chain complex and $\tilde{\xi}$ is a carefully chosen Čech representative for $[\xi] \in H^3(Q;\mathbb{Z})$.

Hence there is a homology spectral sequence with $E^2_{**} = H_*(Q;\mathbb{Z}) \otimes \mathbb{Z}[T,T^{-1}]$ converging to $H_*^{\xi}(Q)$.

(b) $\overline{HM}_*(Y,\mathfrak{s})$ is naturally isomorphic to $H_*^{\xi_3}(\mathbb{T})$.

(c) The spectral sequence for $H_*^{\xi_3}(\mathbb{T})$ has E_∞-term

$$\mathrm{H}_*(\Lambda^* H^1(Y;\mathbb{Z}),\beta) \otimes \mathbb{Z}[T,T^{-1}] \cong \mathrm{gr}\,(\overline{HM}_*(Y,\mathfrak{s})).$$

What is coupled homology, really?

Does K^1 play a fundamental role? Is it only the image of the Chern class map $K^1 \to H^{odd}$ that matters?

Recall that twists of K-theory come about because $PU(H)$ acts on the classifying space Fred(H). A bundle $E \to X$ of projective Hilbert spaces with structure group $PU(H)$ has an associated bundle F_E with fibre Fred(H), and $K^*_E(X)$ is the group $\pi_0\Gamma(F_E)$.

Such bundles $E \to X$ are classified by $[X, BPU(H)]$. By Kuiper's theorem, $BPU(H)$ is a $K(\mathbb{Z}, 3)$. So the twists are parametrised by $[X, BPU(H)] = H^3(X;\mathbb{Z})$.

One (far-fetched) possibility is that coupled cohomology arises similarly, via an action on a product of Eilenberg-MacLane spaces by some group G with $BG \simeq U(\infty)$, for instance Segal's group $G = U_{res}(H)$.