

Lagrangian correspondences and invariants for 3-manifolds with boundary

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MSRI, March 17, 2010

Extending Seiberg–Witten theory

- KRONHEIMER–MROWKA have built a Seiberg–Witten TQFT in $(3 + 1)$ dimensions. The S-W invariants of closed 4-manifolds with $b^+ > 0$ come from a secondary operation which is not part of the TQFT itself.
- An old idea: extend Seiberg–Witten *down* to a TQFT in $(2 + 1 + 1)$ dimensions.
- The invariant of a surface S should be a category built from possible boundary-values of Seiberg–Witten fields on 3-manifolds bounding S .

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Boundary conditions

- Say $\partial W^3 = S_1 \cup S_2$. Choose a Spin^c -structure on W . Define d_i by $c_1(\mathbb{S}^+)[S_i] = 2[1 - g(S_i) + d_i]$.
- The limiting values of S-W fields over the cylindrical completion \hat{W} are *vortices* of degree d_i over S_i .
- The space of vortices over $S_1 \cup S_2$, modulo gauge, is a Kähler manifold identified with

$$\text{Sym}^{d_1}(S_1)_- \times \text{Sym}^{d_2}(S_2).$$

- The limiting values of S-W fields on \hat{W} define a Lagrangian immersion into the space of vortices.

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Our work

- Traditional proposal: attach to Σ_g the Fukaya category $\mathcal{F}(\mathrm{Sym}^{g+k}(\Sigma_g))$.
- To get a reasonable invariant for the 3-ball, take $k = 0$.
- We're working on implementing the $(2 + 1)$ -dimensional aspects of this proposal.
- We use a symplectic (Heegaard Floer) model involving $\mathrm{Sym}^g(\Sigma_g \setminus z)$. No gauge theory.
- AUROUX has found a $(\mathcal{F}\mathrm{Sym}^g(\Sigma_g \setminus z), \mathcal{A}(\Sigma_g, z))$ -bimodule, where $\mathcal{A}(\Sigma_g, z)$ is the bordered Heegaard Floer algebra.
Conjecture: this bimodule induces a quasi-equivalence of module categories under which the two TQFTs coincide.

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Ingredients:

- Balanced Fukaya A_∞ -categories (technology at the level of SEIDEL's book)
- Morse functions with indefinite critical points
- Sequences of Lagrangian correspondences associated with those Morse functions
- Analysis of geometric effect of composition of correspondences (low-tech but takes much of the work)
- Quilted Floer theory: MA'U–WEHRHEIM–WOODWARD A_∞ -bimodules from correspondences
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The target 2-category

A $(2 + 1 + 1)$ -dimensional TQFT is a 2-functor from a cobordism 2-category of 2-, 3- and 4-manifolds to some target 2-category.

Our target 2-category \mathbf{A}_∞ :

- Objects are A_∞ -categories \mathcal{A} (small, coh.-unital, mod 2 graded, linear over a base field \mathbb{K}).
- The $(\mathbb{K}$ -linear) category of 1-morphisms $\mathcal{A} \rightarrow \mathcal{B}$ is the cohomology category $H^0(\mathcal{A} \text{ bimod } \mathcal{B})$ of the dg category $\mathcal{A} \text{ bimod } \mathcal{B}$ of $(\mathcal{A}, \mathcal{B})$ -bimodules.
- Composition

$$H^0(\mathcal{B} \text{ bimod } \mathcal{C}) \otimes H^0(\mathcal{A} \text{ bimod } \mathcal{B}) \rightarrow H^0(\mathcal{A} \text{ bimod } \mathcal{C})$$

is tensor product $(\mathcal{N}, \mathcal{M}) \mapsto \mathcal{N} \otimes_{\mathcal{B}} \mathcal{M}$.

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The cobordism category

- *Ideally*, our TQFT would be a 2-functor

$$Z: \mathbf{Cob}_{2+1+1} \rightarrow \mathbf{A}_\infty$$

defined on the 2-category whose objects are closed, oriented surfaces S , and whose category of 1-morphisms $S_0 \rightarrow S_1$ is the category of closed, oriented cobordisms $S_0 \rightsquigarrow S_1$.

- *Realistically*, we construct a 2-category \mathbf{Cob}^+ and a diagram

$$\begin{array}{ccc} \mathbf{Cob}^+ & \xrightarrow{Z} & \mathbf{A}_\infty \\ \downarrow U & & \\ \mathbf{Cob}_{2+1+1} & & \end{array}$$

The object $Z(\tilde{S})$ depends only on $S = U(\tilde{S})$, up to coherent q.-iso. The 1-morphism $Z(\tilde{Y})$ depends only on $Y = U(\tilde{Y})$ up to q.-iso.

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Constructing \mathbf{Cob}^+

- *Bordered Heegaard theory*
(LIPSHITZ–OZSVÁTH–THURSTON), to 0th approx:
 - Objects are closed oriented surfaces with self-indexing Morse functions.
 - 1-morphisms are 3-manifolds equipped with s.-i. Morse functions extending those on the boundary.
- *Quilted Heegaard theory*
(LEKILI–P.), to 1st approx:
 - Objects are closed, connected surfaces S with basepoint z .
 - 1-morphisms $(S_0, z_0) \rightarrow (S_1, z_1)$ are compact, oriented 3-manifolds Y bounding $\bar{S}_0 \amalg S_1$ with a section z_t of f running from z_0 to z_1 and an *indefinite Morse function* f : critical points of index 1 or 2, connected fibres, minimum fibre S_0 , maximum fibre S_1 , injective on critical set.

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Invariants for based surfaces

- If (S, z) is an oriented surface with basepoint, we will assign to it an A_∞ -category $\mathcal{Z}(S, z)$, 'the' Fukaya category of $\text{Sym}^g(S \setminus \{z\})$.
- This definite article is misleading. We need to specify a symplectic structure and a *balancing form*. Then, after choosing a gigantic collection of perturbation data, we can form the *balanced Fukaya category*.
- Since additional choices (from highly connected spaces) are involved, the objects of \mathbf{Cob}^+ involve more data than I specified previously.

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Symmetric products symplectically

- We need to say how to make $\mathrm{Sym}^g(S)$ symplectic. There are two determining factors in our approach:
 - Our 3-manifold invariants are based on Lagrangian correspondences. We construct these for Kähler forms in specific cohomology classes.
 - In $M = \mathrm{Sym}^g(S \setminus z)$, we want a Fukaya category defined over any ground field (no Novikov rings).
- We work with Kähler forms that arise as the curvature of a *chosen* connection in a certain line bundle $\mathcal{E} \rightarrow \mathrm{Sym}^g(S)$. One has

$$\mathcal{E}|_M \cong \mathcal{K}_M.$$

- The symplectic structure on M is *balanced*, i.e., given as the curvature of a connection in the canonical line bundle.

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Balanced Lagrangians

- Given a Lagrangian submanifold $L \subset M$, let $\sigma: L \rightarrow S\mathcal{K}|_L$ be a complexified orientation form on L . Then the 1-form

$$\sigma^*(-2\pi i\alpha) \in \Omega^1(L)$$

is closed (its derivative is $\omega|_L = 0$). Say L is *balanced* if this form is exact.

- Deformations through balanced Lagrangians are Hamiltonian.

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The balanced Fukaya category

- Notes for Floer-theorists: M is aspherical and has the spherically monotone compactification $\text{Sym}^g(S)$.
- Oriented, balanced Lagrangians are the objects in the *balanced Fukaya category* $\mathcal{F}(M)$, which is set up in the same way as the Fukaya category of an *exact* symplectic manifold.
- Balancing implies that $CF^*(L_0, L_1)$ and the composition maps are defined over the base field \mathbb{K} : there are no periods.
Imprecise conjecture: admissible pairs of Heegaard tori can be understood as balanced pairs of Lagrangians.
- We define $\mathcal{Z}(S, z) = \mathcal{F}(M)$.

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- To an elementary cobordism U from a genus g surface S_0 to a genus $g + 1$ surface S_1 , we can assign to U a Lagrangian correspondence V_U from $\text{Sym}^g(S_0)$ to $\text{Sym}^{g+1}(S_1)$. It embeds into $\text{Sym}^{g+1}(S_1)$ and is a trivial circle-bundle over $\text{Sym}^g(S_0)$.
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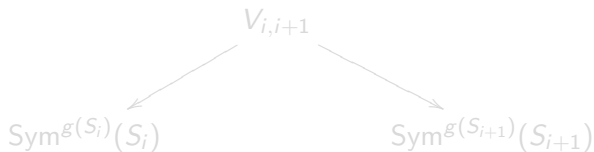
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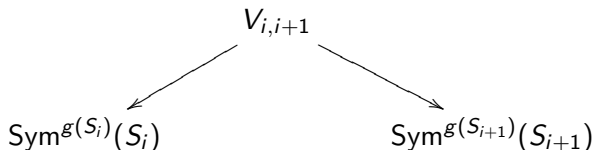
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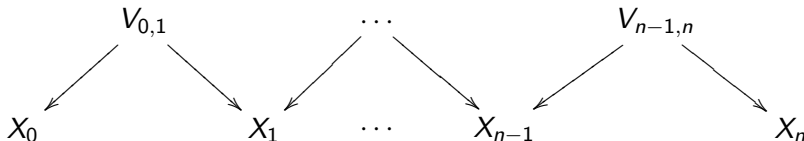


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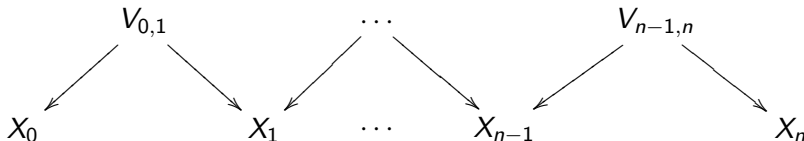
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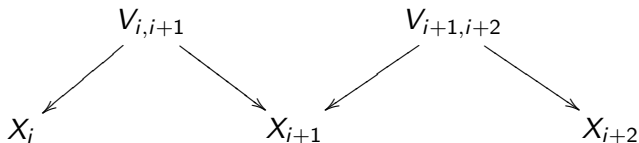


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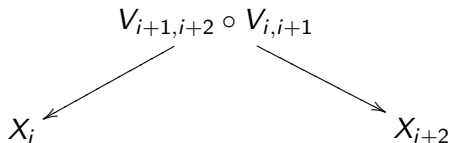
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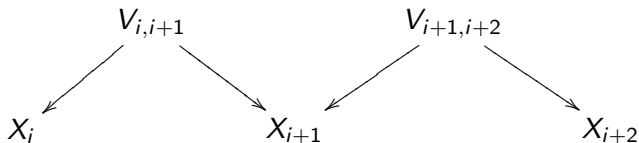


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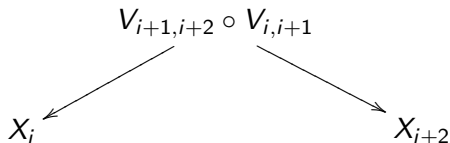
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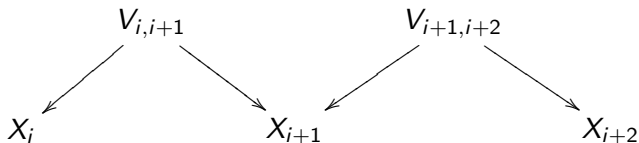


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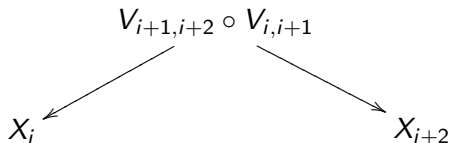
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Handle cancellation

- Suppose α and β are curves in S_1 meeting transversely at a point. Suppose we have elementary cobordisms $U_0: S_0 \rightsquigarrow S_1$ and $U_1: S_1 \rightsquigarrow S_2$ so that α collapses in U_0 , β in U_1 .
- Then the composite $V_{01} \circ V_{12}$ is Hamiltonian isotopic to the diagonal in X_0 .
- This is obvious when $g(S_1) = 1$ (and $g(S_0) = g(S_2) = 0$). It's also then true for the correspondences from $\text{Sym}^n(S_0)$ to $\text{Sym}^{n+1}(S_1)$ to $\text{Sym}^n(S_2)$ obtained by regarding these as symplectic projective vector bundles over $\text{Jac}(S_i)$. We use this to prove it in arbitrary genus (by splitting off a torus summand in S_1).

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- The bimodule $\mathcal{Z}(Y, f, z_t)$ for a 3-manifold assigns to a pair of Lagrangians $L_i \in \text{Ob } \mathcal{Z}(S, z) \times \text{Ob } \mathcal{Z}(S', z')$ ($i = 0, 1$) the *quilted Floer cochains*

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taken in the sequence of balanced manifolds M_i defined by the given path $z_t \subset Y$.

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- The invariant of a (twice punctured) closed 3-manifold Y is a chain complex $QC^*(Y, f)$. Up to quasi-isomorphism, it's independent of f . Taking f self-indexing, we have $QC^*(Y, f) \simeq \widehat{CF}^*(Y, f)$, the Heegaard Floer chains.
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Folds and cusps

- The 2-morphisms (X, F, ζ) in our cobordism category are 4-manifolds with corners X , equipped with

$$F: X \rightarrow [a, b] \times [c, d]$$

and a section ζ of F . Here F should have connected fibres and only indefinite folds and cusps as singularities.

- \mathcal{Z} should attach to (X, F, ζ) a version of the ‘Lagrangian matching invariants’ that I previously defined.
(However, to get an interesting maps, we should enhance \mathcal{Z} to the filtered Fukaya category of $\text{Sym}^g(\Sigma)$, whose hom spaces are set up like CF^+ in Heegaard theory.)

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Manufacturing an algebraic model

- Another natural aim is to manufacture an algebraic model of the theory.
- Compare Khovanov's construction of an extended TQFT for tangles.
- Khovanov's raw material is the Frobenius algebra $H^*(S^2)$. Perhaps ours should be the A_∞ -category $\mathcal{Z}(T^2, z)$.