Lagrangian correspondences and invariants for 3-manifolds with boundary

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Extending Seiberg–Witten theory

- Kronheimer–Mrowka have built a Seiberg–Witten TQFT in $(3+1)$ dimensions. The S-W invariants of closed 4-manifolds with $b^+ > 0$ come from a secondary operation which is not part of the TQFT itself.
- An old idea: extend Seiberg–Witten *down* to a TQFT in $(2+1+1)$ dimensions.
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Boundary conditions

- Say $\partial W^3 = S_1 \cup S_2$. Choose a Spin^c-structure on W. Define d_i by $c_1(\mathbb{S}^+)[S_i] = 2[1 - g(S_i) + d_i]$.
- The limiting values of S-W fields over the cylindrical completion $\hat{W}$ are *vortices* of degree d_i over S_i.
- The space of vortices over $S_1 \cup S_2$, modulo gauge, is a Kähler manifold identified with

$$\mathrm{Sym}^{d_1}(S_1)_- \times \mathrm{Sym}^{d_2}(S_2).$$

- The limiting values of S-W fields on $\hat{W}$ define a Lagrangian immersion into the space of vortices.

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Our work

- Traditional proposal: attach to Σ_g the Fukaya category $\mathcal{F}(\mathrm{Sym}^{g+k}(\Sigma_g))$.
- To get a reasonable invariant for the 3-ball, take $k = 0$.
- We're working on implementing the $(2+1)$-dimensional aspects of this proposal.
- We use a symplectic (Heegaard Floer) model involving $\mathrm{Sym}^g(\Sigma_g \setminus z)$. No gauge theory.
- AUROUX has found a $(\mathcal{F}\,\mathrm{Sym}^g(\Sigma_g \setminus z), \mathcal{A}(\Sigma_g, z))$-bimodule, where $\mathcal{A}(\Sigma_g, z)$ is the bordered Heegaard Floer algebra. *Conjecture*: this bimodule induces a quasi-equivalence of module categories under which the two TQFTs coincide.

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Ingredients:

- Balanced Fukaya A_∞-categories (technology at the level of SEIDEL's book)
- Morse functions with indefinite critical points
- Sequences of Lagrangian correspondences associated with those Morse functions
- Analysis of geometric effect of composition of correspondences (low-tech but takes much of the work)
- Quilted Floer theory: MA'U–WEHRHEIM–WOODWARD A_∞-bimodules from correspondences
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The target 2-category

A $(2+1+1)$-dimensional TQFT is a 2-functor from a cobordism 2-category of 2-, 3- and 4-manifolds to some target 2-category. Our target 2-category $\mathbf{A}_\infty$:

- Objects are A_∞-categories $\mathcal{A}$ (small, coh.-unital, mod 2 graded, linear over a base field $\mathbb{K}$).
- The ($\mathbb{K}$-linear) category of 1-morphisms $\mathcal{A} \to \mathcal{B}$ is the cohomology category $H^0(\mathcal{A}\ \mathit{bimod}\ \mathcal{B})$ of the dg category $\mathcal{A}\ \mathit{bimod}\ \mathcal{B}$ of $(\mathcal{A}, \mathcal{B})$-bimodules.
- Composition

$$H^0(\mathcal{B}\ \mathit{bimod}\ \mathcal{C}) \otimes H^0(\mathcal{A}\ \mathit{bimod}\ \mathcal{B}) \to H^0(\mathcal{A}\ \mathit{bimod}\ \mathcal{C})$$

is tensor product $(\mathcal{N}, \mathcal{M}) \mapsto \mathcal{N} \otimes_{\mathcal{B}} \mathcal{M}$.

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The cobordism category

- *Ideally*, our TQFT would be a 2-functor

$$Z\colon \mathbf{Cob}_{2+1+1} \to \mathbf{A}_\infty$$

defined on the 2-category whose objects are closed, oriented surfaces S, and whose category of 1-morphisms $S_0 \to S_1$ is the category of closed, oriented cobordisms $S_0 \rightsquigarrow S_1$.

- *Realistically*, we construct a 2-category $\mathbf{Cob}^+$ and a diagram

$$\begin{array}{ccc} \mathbf{Cob}^+ & \xrightarrow[Z]{} & \mathbf{A}_\infty \\ \downarrow U & & \\ \mathbf{Cob}_{2+1+1} & & \end{array}$$

The object $Z(\tilde{S})$ depends only on $S = U(\tilde{S})$, up to coherent q.-iso. The 1-morphism $Z(\tilde{Y})$ depends only on $Y = U(\tilde{Y})$ up to q.-iso.

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Constructing **Cob**$^+$

- *Bordered Heegaard theory* (Lipshitz–Ozsváth–Thurston), to 0th approx:
  - Objects are closed oriented surfaces with self-indexing Morse functions.
  - 1-morphisms are 3-manifolds equipped with s.-i. Morse functions extending those on the boundary.
- *Quilted Heegaard theory* (Lekili–P.), to 1st approx:
  - Objects are closed, connected surfaces S with basepoint z.
  - 1-morphisms $(S_0, z_0) \to (S_1, z_1)$ are compact, oriented 3-manifolds Y bounding $\bar{S}_0 \amalg S_1$ with a section z_t of f running from z_0 to z_1 and an *indefinite Morse function* f: critical points of index 1 or 2, connected fibres, minimum fibre S_0, maximum fibre S_1, injective on critical set.

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Invariants for based surfaces

- If (S, z) is an oriented surface with basepoint, we will assign to it an A_∞-category $\mathcal{Z}(S, z)$, 'the' Fukaya category of $\mathrm{Sym}^g(S \setminus \{z\})$.
- This definite article is misleading. We need to specify a symplectic structure and a *balancing form*. Then, after choosing a gigantic collection of perturbation data, we can form the *balanced Fukaya category*.
- Since additional choices (from highly connected spaces) are involved, the objects of $\mathbf{Cob}^+$ involve more data than I specified previously.

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Symmetric products symplectically

- We need to say how to make $\mathrm{Sym}^g(S)$ symplectic. There are two determining factors in our approach:
  - Our 3-manifold invariants are based on Lagrangian correspondences. We construct these for Kähler forms in specific cohomology classes.
  - In $M = \mathrm{Sym}^g(S \setminus z)$, we want a Fukaya category defined over any ground field (no Novikov rings).
- We work with Kähler forms that arise as the curvature of a *chosen* connection in a certain line bundle $\mathcal{E} \to \mathrm{Sym}^g(S)$. One has

$$\mathcal{E}|_M \cong \mathcal{K}_M.$$

- The symplectic structure on M is *balanced*, i.e., given as the curvature of a connection in the canonical line bundle.

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Balanced Lagrangians

- Given a Lagrangian submanifold $L \subset M$, let $\sigma\colon L \to S\mathcal{K}|_L$ be a complexified orientation form on L. Then the 1-form

$$\sigma^*(-2\pi i\alpha) \in \Omega^1(L)$$

is closed (its derivative is $\omega|_L = 0$). Say L is *balanced* if this form is exact.

- Deformations through balanced Lagrangians are Hamiltonian.

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The balanced Fukaya category

- Notes for Floer-theorists: M is aspherical and has the spherically monotone compactification $\mathrm{Sym}^g(S)$.
- Oriented, balanced Lagrangians are the objects in the *balanced Fukaya category* $\mathcal{F}(M)$, which is set up in the same way as the Fukaya category of an *exact* symplectic manifold.
- Balancing implies that $CF^*(L_0, L_1)$ and the composition maps are defined over the base field $\mathbb{K}$: there are no periods. *Imprecise conjecture*: admissible pairs of Heegaard tori can be understood as balanced pairs of Lagrangians.
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- Notes for Floer-theorists: M is aspherical and has the spherically monotone compactification $\mathrm{Sym}^g(S)$.
- Oriented, balanced Lagrangians are the objects in the *balanced Fukaya category* $\mathcal{F}(M)$, which is set up in the same way as the Fukaya category of an *exact* symplectic manifold.
- Balancing implies that $CF^*(L_0, L_1)$ and the composition maps are defined over the base field $\mathbb{K}$: there are no periods. *Imprecise conjecture*: admissible pairs of Heegaard tori can be understood as balanced pairs of Lagrangians.
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Isn't that an unknowable invariant?

- I expect we shall never know all the objects in $\mathcal{F}(M)$.
- AUROUX observed that, by SEIDEL's generation theorem, the triangulated envelope $Tw\,\mathcal{F}(M)$ embeds into a directed A_∞-category with $\binom{2g}{g}$ objects (non-compact Lagrangian thimbles).
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Elementary cobordisms

- To an elementary cobordism U from a genus g surface S_0 to a genus $g+1$ surface S_1, we can assign to U a Lagrangian correspondence V_U from $\mathrm{Sym}^g(S_0)$ to $\mathrm{Sym}^{g+1}(S_1)$. It embeds into $\mathrm{Sym}^{g+1}(S_1)$ and is a trivial circle-bundle over $\mathrm{Sym}^g(S_0)$.
- The construction is from my thesis. It uses algebro-geometric methods (degenerations of symmetric products).
- Conjecture: up to isotopy, V_U is the locus of pairs of solutions to the abelian vortex equations over S_0, S_1 which are common asymptotic limits of solutions to the (perturbed) SW equations over $\hat{U} = U \cup (\text{cyl. ends})$.

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Indefinite Morse functions

- A choice of indefinite Morse function f on a cobordism $Y : S \rightsquigarrow S'$ breaks it into elementary cobordisms

$$S = S_0 \overset{U_0}{\rightsquigarrow} S_1 \overset{U_1}{\rightsquigarrow} S_2 \cdots \overset{U_{n-1}}{\rightsquigarrow} S_n = S'$$

  where $g(S_{i+1}) = g(S_i) \pm 1$.

- We can attach to (Y, f) a chain $V_{(Y,f)}$ of Lagrangian correspondences

$$\begin{array}{ccc} & V_{i,i+1} & \\ \swarrow & & \searrow \\ \mathrm{Sym}^{g(S_i)}(S_i) & & \mathrm{Sym}^{g(S_{i+1})}(S_{i+1}) \end{array}$$

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Indefinite Morse functions, contd.

Let $X_i = \mathrm{Sym}^{g(S_i)}(S_i)$.

- Up to symplectic isotopies of the factors, the chain $V_{Y,f}$ given by

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- We prove that the *Wehrheim–Woodward equivalence class* of $V_{Y,f}$ independent of f.

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Equivalent sequences of correspondences

- A pair of correspondences

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Handle cancellation

- Suppose α and β are curves in S_1 meeting transversely at a point. Suppose we have elementary cobordisms $U_0 \colon S_0 \rightsquigarrow S_1$ and $U_1 \colon S_1 \rightsquigarrow S_2$ so that α collapses in U_0, β in U_1.
- Then the composite $V_{01} \circ V_{12}$ is Hamiltonian isotopic to the diagonal in X_0.
- This is obvious when $g(S_1) = 1$ (and $g(S_0) = g(S_2) = 0$). It's also then true for the correspondences from $\mathrm{Sym}^n(S_0)$ to $\mathrm{Sym}^{n+1}(S_1)$ to $\mathrm{Sym}^n(S_2)$ obtained by regarding these as symplectic projective vector bundles over $Jac(S_i)$. We use this to prove it in arbitrary genus (by splitting off a torus summand in S_1).

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Quilted Floer cohomology

- The bimodule $\mathcal{Z}(Y, f, z_t)$ for a 3-manifold assigns to a pair of Lagrangians $L_i \in Ob\,\mathcal{Z}(S, z) \times Ob\,\mathcal{Z}(S', z')$ $(i = 0, 1)$ the *quilted Floer cochains*

$$QC^*(L_0, V_{Y,f}, L_1)$$

  taken in the sequence of balanced manifolds M_i defined by the given path $z_t \subset Y$.
- Ma'u and Wehrheim–Woodward: this procedure defines an A_∞-bimodule. Concatenating sequences (as in composition of cobordisms) gives tensor product of bimodules.
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Closed 3-manifolds

- We have $\mathcal{Z}(S^2, z) = *$, the trivial A_∞-category whose only object $*$ has $hom(*, *) = \mathbb{K}$. A $(*, *)$-bimodule is just a chain complex.
- The invariant of a (twice punctured) closed 3-manifold Y is a chain complex $QC^*(Y, f)$. Up to quasi-isomorphism, it's independent of f. Taking f self-indexing, we have $QC^*(Y, f) \simeq \widehat{CF}^*(Y, f)$, the Heegaard Floer chains.
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Folds and cusps

- The 2-morphisms (X, F, ζ) in our cobordism category are 4-manifolds with corners X, equipped with

$$F\colon X \to [a, b] \times [c, d]$$

and a section ζ of F. Here F should have connected fibres and only indefinite folds and cusps as singularities.

- $\mathcal{Z}$ should attach to (X, F, ζ) a version of the 'Lagrangian matching invariants' that I previously defined.
(However, to get an interesting maps, we should enhance $\mathcal{Z}$ to the filtered Fukaya category of $\mathrm{Sym}^g(\Sigma)$, whose hom spaces are set up like CF^+ in Heegaard theory.)

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Manufacturing an algebraic model

- Another natural aim is to manufacture an algebraic model of the theory.
- Compare Khovanov's construction of an extended TQFT for tangles.
- Khovanov's raw material is the Frobenius algebra $H^*(S^2)$. Perhaps ours should be the A_∞-category $\mathcal{Z}(T^2, z)$.