Fibered 3-manifolds and the Floer homology of fibered Dehn twists

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Fibered 3- and 4-manifolds

I'll tell you about a fragment of an ongoing project to build and study a symplectic analogue of Seiberg-Witten theory in 3 and 4-dimensions in the presence of singular surface-fibrations, i.e.

(i) 'Harmonic' S^1-valued Morse functions on 3-manifolds.

(ii) 'Broken fibrations' on 4-manifolds.

Study pseudo-holomorphic curves in associated fibrations by symmetric products, subject to Lagrangian boundary conditions.

Singular fibrations

A 'harmonic' Morse function on Y^3 is a function $Y^3 \to S^1$, each of whose critical points is non-degenerate of index 1 or 2.

Models: $(x_1, x_2, x_3) \mapsto \pm(x_1^2 + x_2^2 - x_3^2)$.

Say it's also injective on its critical set.

A broken fibration on X^4 is a map $\pi\colon X \to S$ onto a surface, injective on its critical set, with two types of critical points:

isolated: $(z_1, z_2) \mapsto z_1 z_2$,

non-isolated: $(t, x_1, x_2, x_3) \mapsto (t, \pm(x_1^2 + x_2^2 - x_3^2))$.

There's a cobordism category in which the objects are 3-manifolds with harmonic Morse functions and the morphisms are 4-manifolds with broken fibrations.

Dehn twists

Today I will discuss Floer homology for 3-manifolds fibered over S^1. I will focus on mapping tori of *reducible* surface-diffeomorphisms—in particular, right-handed Dehn twists.

Key points:

(i) Lefschetz fibrations play a lead role.

More surprisingly, our treatment of reducible diffeomorphisms also brings in broken fibrations.

(ii) For the Dehn twist, Floer homology will be expressed as a classical relative homology group.

There is enough structure here that one can start to contemplate arbitrary compositions of positive Dehn twists.

Note: this is work in progress.

Floer homology for symplectic automorphisms

This version of Floer homology attaches a Λ_R-module

$$HF_*(E,\Omega) = \mathsf{H}(CF_*(E,\Omega),\delta_J).$$

to every fiber bundle $E \to S^1$ equipped with a closed, fiberwise non-degenerate 2-form Ω.

We assume that the symplectic fibers are compact, 'weakly monotone' symplectic manifolds.

Λ_R is the universal Novikov ring of the commutative ring R. We can take $R = \mathbb{Z}$.

It's invariant under $\Omega \rightsquigarrow \Omega + d\alpha$ where α vanishes on the fibers.

Defining Floer homology

$CF_*(E, \Omega)$ is the free Λ_R-module on the set of horizontal sections of $E \to S^1$ (tangent to $\ker \Omega$).

The differentials are defined via finite-energy J-holomorphic sections of $\mathbb{R} \times E \to \mathbb{R} \times S^1$, where J is an $\mathbb{R}$-invariant almost complex structure sending $\frac{\partial}{\partial t}$ to a horizontal vector field.

Correspondence:

$$(E \to S^1, \Omega) \leftrightarrow (M, \omega, \mu) \text{ where } \mu \in \mathrm{Aut}(M, \omega).$$

Fibered 3-manifolds

Consider a fibered 3-manifold $p\colon Y^3 \to S^1$ and a class $w \in H^2(Y;\mathbb{R})$ with $w(\text{fiber}) > 0$.

Choose a vertical complex structure j on Y. It makes $Y^{[n]} := \{(D,t) : t \in S^1, D \in \mathrm{Sym}^n(Y_t)\}$ a smooth manifold, fibered over S^1. The fibers are complex manifolds.

Choose a closed 2-form $\Omega^{[n]}$ from the convex set of forms s.t.

- $\Omega^{[n]}$ is fiberwise-Kähler;

- $\Omega^{[n]}$ represents a certain cohomology class, determined by w via a 'mu-map'.

Define $HF_*(Y,p,n,w) := HF_*(Y^{[n]},\Omega^{[n]})$.

This is a symplectic cousin of the monopole Floer homology

$$\bigoplus_{\langle c_1(\mathfrak{s}),[\Sigma]\rangle=2(n+1-g)} HM_\bullet(Y,\mathfrak{s};w).$$

When $Y = T_\phi$ for $\phi \in \mathrm{Diff}(\Sigma, \mathrm{area})$, write $HF_*(\phi, n)$ for $HF_*(Y, p, n, w)$.

When $n > 1$, there are almost no computations besides the identity map:

$$\begin{aligned} HF_*(\mathrm{id}, n) &\cong H^*(\mathrm{Sym}^n(\Sigma)) \\ &\cong \bigoplus_{i=0}^{n} \{0, 2, \ldots, 2i\}\Lambda^{n-i}H^1(\Sigma). \end{aligned}$$

I'm working on machinery which will compute these groups for e.g. a right-handed Dehn twist.

The case $n = 1$, and many of the ideas, are due to Paul Seidel.

Floer homology for symmetric products of Dehn twists

Let $\gamma \subset \Sigma$ be an embedded circle. Fix $n > 0$.

Theorem. We can associate with γ a Hamiltonian isotopy-class of hypersurfaces $V_\gamma \subset \mathrm{Sym}^n(\Sigma)$ diffeomorphic to the total space of an S^1-bundle over $\mathrm{Sym}^{n-1}(\Sigma_\gamma)$, where Σ_γ is the surface obtained by surgery along γ.

V_γ is obtained as a vanishing cycle for a degeneration of $\mathrm{Sym}^n(\Sigma)$. Its uniqueness follows from the observation of W. Ruan that deformations of fibered coisotropic submanifolds, like those of Lagrangians, are governed by a 'flux' class.

Conjecture:

Soon a theorem for restricted n?

Let τ_γ be the right-handed Dehn twist about γ. Then

$$HF_*(\tau_\gamma, n) \cong H_*(\mathrm{Sym}^n(\Sigma), V_\gamma; \Lambda_{\mathbb{Z}}).$$

When γ is non-separating, the S^1-bundle is trivial, and $H_*(V_\gamma) \to H_*(\mathrm{Sym}^n(\Sigma))$ injective. Calculate that $H_*(\mathrm{Sym}^n(\Sigma), V_\gamma)$ is

$$\Lambda^n H^1(\Sigma_\gamma) \oplus \sum_{i=1}^{n} \{1, 2, \ldots, 2i\} \Lambda^{n-i} H^1(\Sigma_\gamma)$$

This is consistent with Hutchings & Sullivan's periodic Floer homology calculation.

Outline of the proof

(1) τ_γ acts on $\mathrm{Sym}^n(\Sigma)$ as the symplectic monodromy of a 'holomorphic Morse-Bott' family over the disc.

(2) The monodromy is Hamiltonian-isotopic to a fibered Dehn twist about a vanishing cycle $V_\gamma \subset \mathrm{Sym}^n(\Sigma)$ which is S^1-fibered over $\mathrm{Sym}^{n-1}(\Sigma_\gamma)$.

(3) A long exact sequence describes the effect of fibered Dehn twists on Floer homology.

LES for fibered Dehn twists

Let V be a submanifold of (M, ω_M) which is also the total space of an orientable S^k-bundle $\rho\colon V \to N$ over a symplectic manifold (N, ω_N) such that $\omega_M|_V = \rho^*\omega_N$.

Conjecture (Seidel): *There's a long exact sequence*

$$\begin{tikzcd} HF_*^{M\times N_-}(\widehat{V}, (\mu \times 1)\widehat{V}) \arrow[r] & HF_*(\mu) \arrow[r] & HF_*(\mu \circ \tau_V) \arrow[ll, bend left] \end{tikzcd}$$

where $\widehat{V} = graph(\rho) \subset M \times N_-$.

When $M = \mathrm{Sym}^n(\Sigma)$ and $V = V_\gamma$, this is a translation of the surgery sequence in monopole Floer homology.

Gysin sequence

Suppose $\mu \in \mathrm{Aut}(M, \omega_M)$ preserves V and covers $\nu \in \mathrm{Aut}(N, \omega_N)$. Then there's an isomorphism

$$HF_*^{M \times N_-}(\widehat{V}, (\mu \times 1)\widehat{V}) \cong \mathrm{Cone}(e)$$

where

$$e = e(V) \cap \cdot : HF_*(\nu) \to HF_{*-k-1}(\nu)$$

is quantum cap product by the Euler class $e(V) \in H^{k+1}(N)$.

This is a symplectic cousin of a formula for 1-handle addition in monopole Floer homology.

Reducible surface-diffeomorphisms

Suppose $\phi \in \mathrm{Diff}^+(\Sigma)$ preserves a circle γ and induces $\phi_\gamma \in \mathrm{Diff}^+(\Sigma_\gamma)$. We can then combine the fibered Dehn twist (surgery) sequence with the Gysin (1-handle) sequence.

This results in an exact triangle

$$
\begin{tikzcd}
HF_*(\phi, n) \arrow[rr] & & HF_*(\phi \circ \tau_\gamma, n) \arrow[dl] \\
& HF_*(\phi_\gamma, n-1) \otimes H_*(S^1) \arrow[ul] &
\end{tikzcd}
$$

relating elementary Lefschetz fibrations with elementary broken fibrations.