Research Methods in Mathematics
Extended assignment 3

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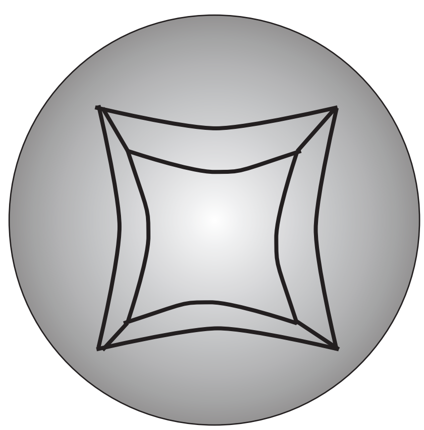
Due: Noon, Tuesday 14 December. You can turn it in at my office, or in my mailbox in the math office on the 8th floor of RLM, or by email.

This is a short assignment but you need to read it carefully so as to understand the concepts.

Let S be the surface of a sphere. Let G be a *graph* inscribed on S. That means that G consists of a finite collection of *vertices* (these are points on S) and a finite collection of *edges*—these are curves joining two vertices. Edges cannot cross over other edges, nor can they cross over themselves. See the example in Figure 1, which has 8 vertices and 12 edges.



The *Euler characteristic* of a graph G is defined as

$$\chi(G) = v - e + f,$$

where v is the number of vertices, e the number of edges, and f the number of faces (the faces are the regions of S left behind when you delete the graph; there are six in the example). Euler's theorem says that, providing that G is *connected* and has at least one vertex, one has $\chi(G) = 2$. Connected means that you can walk from one point on the graph to another without leaving it.) For instance, the example shown has $\chi(G) = 8 - 12 + 6 = 2$.

It's important that G is connected. E.g. take G to have two vertices and no edges; then $\chi(G) = 2 - 0 + 1 = 3$.

We'll prove Euler's theorem.

(1) We'll prove the theorem by induction on the number of edges. Start this off by proving that $\chi(G) = 2$ if there are no edges.

(2) Now suppose there's at least one edge. Pick a vertex v of G. If w is another vertex, its *distance* from v is the minimum number of edges in a walk from v to w. Pick a vertex w whose distance from v is *as large as possible.*

   (a) If w is joined to at least two edges, let G' be the graph obtained by deleting one of these edges. Explain why G' is connected.

   (b) Let G' be as in (a). I claim that G' has one fewer faces than G. Explain why this is so by using the *Jordan curve theorem*: any closed, non-self-intersecting loop in S divides S into exactly two regions ('inside' and 'outside' the loop). [This is the tricky point in the proof!]

   (c) Deduce that $\chi(G') = \chi(G)$.

   (d) Now suppose w is joined to only one edge. Let G'' be the graph obtained by deleting w and that edge. Explain why G'' is connected and why $\chi(G'') = \chi(G)$.

(3) Explain how to run the induction.

Look at the wallpaper pattern W on the class webpage which is labeled $*442$. It consists entirely of straight lines. In the top-left of the picture you can a square region subdivided into eight triangles. Call this region Q. The pattern consists of many copies of Q, related to the original by translational symmetries.

(4) Suppose we look at the part of the pattern consisting of an $n \times n$ grid of copies of Q. For instance, the picture on the webpage is the 3×3 case. Regard this part of the pattern as a graph $G(n)$ on a little region of the sphere S.
Show that G has $8n^2 + 1$ faces. Find formulas for the numbers of edges and of vertices and confirm that $\chi(G(n)) = 2$.

Here—just for interest—is what this has to do with Conway's magic formula. This formula says that the cost of a wallpaper orbifold is exactly 2. In this example, the orbifold is the little blue triangle T. Its edge costs 1. A corner angle π/N costs

$(N-1)/2N$, so in this case the corners cost $3/8$, $3/8$ and $1/4$ so the total cost is $1+3/8+3/8+1/4=2$.

Let's look again at our calculation of $\chi(G(n))$. In Q, there are 8 copies of T. That means that in $G(n)$ there are $8n^2$ faces (because we have n^2 copies of T), plus one face outside the pattern. There are $(3\times 8/2)n^2$ edges (because we have eight copies of T, each of which has 3 edges, but each edge is shared between two triangles copies of T) plus some extra edges at the boundary. There are $8n^2(\frac{1}{8}+\frac{1}{8}+\frac{1}{4})$ vertices, plus some at the boundary (because each copy of the bottom-left vertex of T is shared by four triangles, while each copy of the bottom-right or top-left vertex is shared by 8 triangles).

So $\chi(G(n))/n^2 = 8[1-\frac{3}{2}+\frac{1}{8}+\frac{1}{8}+\frac{1}{4})]$, plus some boundary corrections which are small (smaller than a constant times $1/n$). But $\chi(G(n))/n^2 = 2/n^2 \to 0$ as $n\to\infty$. Hence the term in square brackets,

$$1-\frac{3}{2}+\frac{1}{8}+\frac{1}{8}+\frac{1}{4},$$

which doesn't involve n, must be zero. It is, as you can check!

By thinking carefully about how the expression in square brackets generalizes, you find a relation between the features of the orbifold of any orbifold, saying that a certain signed sum is equal to zero. Rearranging this, you get Conway's magic formula.