Research Methods in Mathematics
Homework 10

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Due November 18.

READ ME!! *Terminology:* By the 'diagonal' of a 3×3 matrix, I mean the line running from top-left to bottom-right. The diagonal entries are the ones on this line. A 3×3 matrix is upper-triangular if all entries strictly below the diagonal are zero. It is diagonal if all non-diagonal entries are zero.

(1) *Understanding concepts from class.* In class, we defined the determinant of a 3×3 matrix $A = [\mathbf{c}_1 \mathbf{c}_2 \mathbf{c}_3]$ as

$$\det(A) = (\mathbf{c}_1 \times \mathbf{c}_2) \cdot \mathbf{c}_3.$$

Let

$$\mathbf{c}_i = \begin{bmatrix} c_{1i} \\ c_{2i} \\ c_{3i} \end{bmatrix}, \quad i = 1, 2, 3.$$

(a) Find an expression for $\det(A)$ in terms of the entries c_{ij}.

(b) Use your expression to show that if any two columns of A are exchanged, the effect on the determinant is to multiply it by -1.

(2) *Developing concepts from class.* Show that if a 3×3 matrix A is upper triangular then $\det(A)$ is the product of the diagonal entries.

(3) *Developing concepts from class.* In proving that every 3×3 matrix has an eigenvalue, we needed the following fact: for any $a, b, c \in \mathbb{R}$, the function $f(x) = -x^3 + ax^2 + bx + c$ has at least one root (that is: there is some x_0 so that $f(x_0) = 0$). We now fill in the details. Let $X = \max(1, 4|a|, 4|b|, 4|c|)$.

(a) Show that when $x \geq X$ one has

$$f(x) \leq -\frac{1}{4}x^3 < 0.$$

(b) Show that when $x \leq -X$, one has

$$f(x) \geq -\frac{1}{4}x^3 > 0.$$

(c) Use the intermediate value theorem to deduce that f has a root.

(4) *Understanding concepts from class.* Let

$$X = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 9 \\ 0 & 0 & -2 \end{bmatrix}$$

(a) Find the characteristic polynomial $\chi_X(\lambda) := \det(X - \lambda I)$.

(b) Find the eigenvalues of X (recall that they are the roots of the characteristic polynomial).

(c) For each eigenvalue λ, find an eigenvector, i.e., a non-zero vector $\mathbf{v}$ such that $X\mathbf{v} = \lambda v$. You will need to solve the system of simultaneous linear equations $(X - \lambda I)\mathbf{v} = \mathbf{0}$.

(5) *Understanding concepts from class.* Repeat the previous question for

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

One of the eigenvalues has 'geometric multiplicity 2.' That is, there are two eigenvectors, neither a multiple of the other. Find two such eigenvectors.

(6) *Developing concepts from class.* Suppose that a 3×3 matrix M has 3 eigenvalues λ_1, λ_2 and λ_3, *all different*. Prove that there is an invertible matrix P such that

$$M = P\Lambda P^{-1},$$

where

$$P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

(7) *Problem.* A 3×3 matrix M is called *diagonal* if its non-zero entries all lie on the diagonal. It is called *diagonalizable* there exists an invertible matrix P and a diagonal matrix Λ such $M = P\Lambda P^{-1}$. Find a 3×3 matrix which is *not* diagonalizable.
[Hints: show that the diagonal entries of Λ would have to be eigenvalues of M. Then look for an M such that $M^3 = 0$.]