Research Methods in Mathematics
Homework 2

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Due at the beginning of class, Thursday September 9.

Except in question 4, you should assume all the familiar properties of addition and multiplication of natural numbers.

(1) *Learning methods from class: induction.*

(a) Prove by induction that $1+2+\cdots+n = n(n+1)/2$ for all natural numbers n.

(b) Prove by induction that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all natural numbers n. Hence calculate $1^3 + 2^3 + \cdots + 100^3$.
I don't know of any 'conceptual' (e.g. visual) proof that illuminates why this formula is true. All the same, we can prove it by induction.

(2) *Exploring concepts from class: induction.* The axioms for natural numbers said that we have 1 and the successor $n \mapsto S(n)$. There is no n such that $1 = S(n)$. If $m \neq n$ then $S(m) \neq S(n)$. The principle of induction holds. In class, I mentioned another axiom: that *for any natural number n, we have $S(n) \neq n$.* I slipped up here: the last one is redundant! It follows from the other axioms. Prove this!

(3) *Exploring concepts from class: axioms for natural numbers.*

(a) Suppose we work in clock arithmetic. That is, we work with the number $1, 2, 3, \ldots, 12$ and then consider 13 the same as 1, 14 the same as 2, and so on. This system of numbers has a notion of 1 and a notion of successor $S(n)$ (add 1). Show that these notions do not satisfy the axioms for natural numbers. *Extra credit. Show that no matter how we define 1 and $S(n)$ in this system, the axioms can never be satisfied.*

(b) Now work with the integers (positive, negative and zero). Again, we know what we mean by 1 and by $S(n)$ (add 1). Show that these notions do not satisfy the axioms for natural numbers. *Extra credit. Find a way to redefine 1 and $S(n)$ in such a way that the axioms are satisfied.*

(4) *Exploring concepts from class: addition/induction.* In this question, you should work directly with the definition of addition, which says that $n + 1 = S(n)$ and $n + S(k) = S(n + k)$ (this inductively defines $n + 2$, $n + 3$, etc.). Use it to prove that $1 + n = n + 1$ for all natural numbers n.

(5) *Problem.*

(a) Let $k(n)$ be the number of ways of selecting any number of things from n (disregarding order). For example, $k(2) = 4$, the possibilities being that we choose both things, just the first, just the second, or neither. Prove that $k(n) = 2^n$.

(b) Let $j(n)$ be the number of ways of selecting an odd number of things from n (again disregarding order). For example, $j(2) = 2$, as we must choose just the first thing or just the second. Prove that $j(n) = 2^{n-1}$.