Research Methods in Mathematics
Homework 3

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Due at the beginning of class, Thursday September 16.

(1) *Understanding concepts from class.* Referring to your notes, explain the formal definition of a fraction p/q. (Here p and q are natural numbers.)

(2) *Understanding concepts from class.* What is wrong with the following attempts to define a function f which inputs a fraction and outputs another fraction:

(a) $f(p/q) = (p+1)/(q+1)$.

(b) $f(p/q) = p/q^2$.

If we attempt to define $f(p/q) = p^2/q^2$, is this a legitimate definition or not? Explain.

(3) *Using concepts from class.* Prove that for all fractions a, b and c, we have

$$a(b+c) = ab + ac. \tag{‡}$$

(In proving (‡), you may take for granted the associative, commutative and distributive laws for adding and multiplying natural numbers.)

(4) *Using concepts from class.* Use the definitions of inequality and multiplication to show that $n \times 2 > n$ for all natural numbers n.

(5) *Using concepts from class.* Prove by induction that for every natural number n, the inequality $n > n$ is *false*.

(6) *Understanding concepts from class.* Formulate an inductive definition of the nth power m^n, where m and n are natural numbers.

(7) *Problem.*

(a) Prove that for natural numbers a and b, we have

$$4ab \leq (a+b)^2.$$

[*Hint*: use subtraction; find another way to express $(a+b)^2 - 4ab$.]

(b) Show that if $a = b$ then

$$4ab = (a+b)^2. \tag{⋆}$$

(c) Show conversely that if (⋆) holds for some natural numbers a and b then $a = b$. [*Hint*: analyze your argument from (a).]