Research Methods in Mathematics
Homework 4

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Due at the beginning of class, September 23.

(1) *Using concepts from class.* This question is about consequences of the inequality $|x+y| \leq |x| + |y|$, which we proved in class.

(a) Prove that for all rational a, b and c, the following inequality holds:

$$|a-c| \leq |a-b| + |b-c|.$$

(b) Prove that

$$|a| \leq |b| + |a-b|.$$

(c) Prove that

$$\Big| |a| - |b| \Big| \leq |a-b|.$$

[*Hint*: when $|a| \geq |b|$ you can use part (b).]

(2) *Understanding concepts from class.* Fill in the details of Euclid's famous proof that there is no rational number x such that $x^2 = 2$: We suppose that $x = p/q$, with p and q integers and $q \neq 0$, and that $x^2 = 2$. We will show that this assumption leads to a contradiction.

(a) Show that p^2 must be even.

(b) Show that p must be even.

(c) Show that q^2 must be even.

(d) Show that q must be even.

(e) Obtain a contradiction by writing the fraction in its lowest terms. Explain why this completes the proof.

(3) Which of the following sets of rational numbers has an upper bound? Exhibit an upper bound or show that there isn't one (and explain your answers). Do the same for lower bounds.

(a) The set of all squares $\{0^2, 1^2, 2^2, 3^2, \dots\}$.

(b) The set of all reciprocals of natural numbers, $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.

(c) The set of $x \in \mathbb{Q}$ such that
$$\frac{1}{1+x^2} \leq 3.$$
(d) The set of $x \in \mathbb{Q}$ such that
$$\frac{1}{1+x^2} \geq \frac{1}{3}.$$

(4) Which of the sets in the previous question has a (rational) *least* upper bound? Which has a greatest lower bound? In each case, exhibit a least upper bound (explaining your answer) or show that there isn't one. Do the same for greatest lower bounds.

(5) *Problem.* When we add three natural numbers a, b and c, there are two ways we can do it without changing their order: we can evaluate $(a+b)+c$ or $a+(b+c)$. They give the same result, by associativity. When we add four numbers a, b, c and d, there are five ways to do it without changing the order (i.e., using associativity but not commutativity):
$$((a+b)+c)+d, \quad (a+(b+c))+d,$$
$$(a+b)+(c+d), \quad a+((b+c)+d), \quad a+(b+(c+d)).$$
How many ways are there to add five numbers without changing their order? Can you find a diagrammatic way to represent the possibilities that illuminates what's going on?
Extra credit. The nth Catalan number C_n is defined to be the number of ways of adding $n+1$ numbers. We have $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, and you just calculated C_4. For convenience, we put $C_0 = 1$. Prove that
$$C_{n+1} = C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \cdots + C_nC_0.$$
Use this to calculate C_5.