Research Methods in Mathematics
Homework 5

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Due at the beginning of class, Thursday September 30.

(1) *Understanding concepts from class.*

  (a) Suppose that x is a rational number such that $x^2 < 2$. Show that there exists a rational number $\epsilon > 0$ such that $(x - \epsilon)^2 < 2$.

  (b) Let $a \in \mathbb{R}$ be a real number, $a > 0$. Suppose that x is a real number such that $x^2 < a$. Show that there exists a real number $\epsilon > 0$ such that $(x + \epsilon)^2 < a$.

  (c) Let S_a be the set of reals x such that $x^2 = a$. Show that a has a least upper bound $s_a = \sup S_a$. Prove that $s_a^2 = a$. (So every positive real number has a real square root!)

(2) *Using concepts from class.* Suppose $a \in \mathbb{R}$ with $a > 0$.

  (a) In the last question, we defined a square root

$$a^{1/2} = s_a = \sup\{x \in \mathbb{R} : x^2 < a\}.$$

  How would you go about defining $a^{1/3}$ when a is a positive real? Start by explaining the strategy, and if you can, fill in the details.

  (b) What about $a^{1/n}$ when n is a natural number?

  (c) What about a^b when b is a positive rational?

  (d) What about a^b when b is a positive real?

(3) *Problem.* It is a fact that $(a^b)^c = a^{bc}$ for positive reals a, b, c. This fact can be used to prove that

  $\star$ there exist positive real numbers a and b, neither of them rational, such that a^b is rational.

Consider the possibilities $(a, b) = (\sqrt{2}, \sqrt{2})$ and $(a, b) = (\sqrt{2}^{\sqrt{2}}, b = \sqrt{2})$. Show that one of these pairs has the property $\star$.

Comment: You won't be able to tell which pair actually works! This proof is 'non-constructive', it only proves that a pair exists. It is known by other means that the second pair gives an example and the first pair does not.

(4) *Understanding concepts from class.* The definition of a limit goes like this. Let f be a function whose domain includes all numbers close to a, except perhaps a itself. *Say $f(x) \to L$ as $x \to a$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\text{whenever } 0 < |x - a| < \delta, \text{ we have } |f(x) - L| < \epsilon.$$

Which of the following sentences is equivalent to the statement 'it is not true that $f(x) \to L$ as $x \to a$'?

(a) There is no $\epsilon > 0$ such that there exists $\delta > 0$ for which

$$\text{whenever } 0 < |x - a| < \delta, \text{ we have } |f(x) - L| < \epsilon.$$

(b) There exists $\epsilon > 0$ such that for every $\delta > 0$ there is some x such that

$$0 < |x - a| < \delta$$

and

$$|f(x) - L| < \epsilon.$$

(c) For every $\epsilon > 0$ there is some $\delta > 0$ and some x such that

$$0 < |x - a| < \delta$$

and

$$|f(x) - L| < \epsilon.$$

(d) For all $\delta > 0$, there exists $\epsilon > 0$ such that such that

$$\text{whenever } 0 < |x - a| < \delta, \text{ we have } |f(x) - L| < \epsilon.$$

Explain.

(5) *Understanding concepts from class.* Look at the graphs in Figures 1-5. Explain whether the function shown has a limit L as $x \to a$, and if so, what the value of L is. (Formal proofs aren't required, but clear explanations are.)

(6) *Using concepts from class* Let $\epsilon = 10^{-3}$. Given f, a, L, find a $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

(a) $f(x) = 8x - 9$; $a = 2$; $L = 7$.

(b) $f(x) = x^2$; $a = 1$; $L = 1$.

Now *prove* in each case that $f(x) \to L$ as $x \to a$. (You will have to consider arbitrary $\epsilon > 0$.)

(7) *Using concepts from class.* Let $g(x) = \sin(1/x)$. Sketch the graph of g. Carefully prove that g has *no* limit as $x \to 0$.

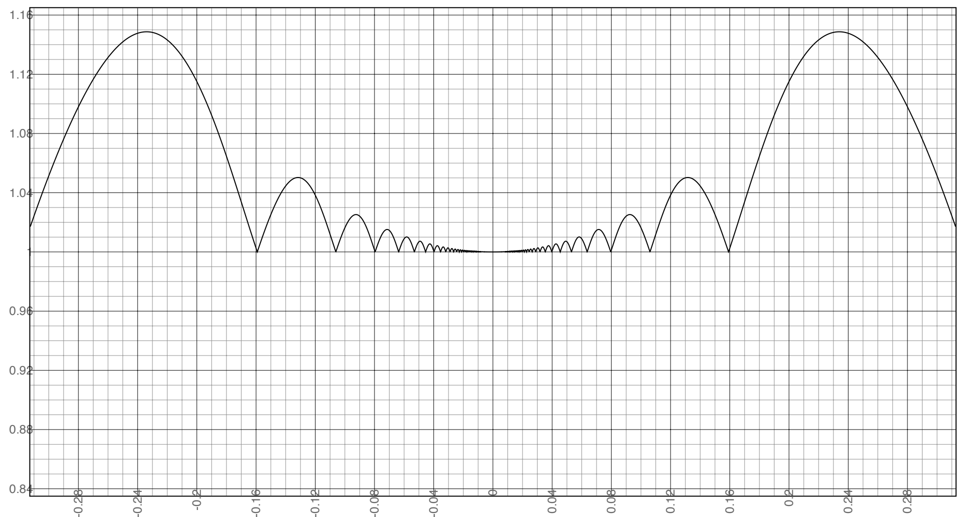

Figure 1: $a = 0$.

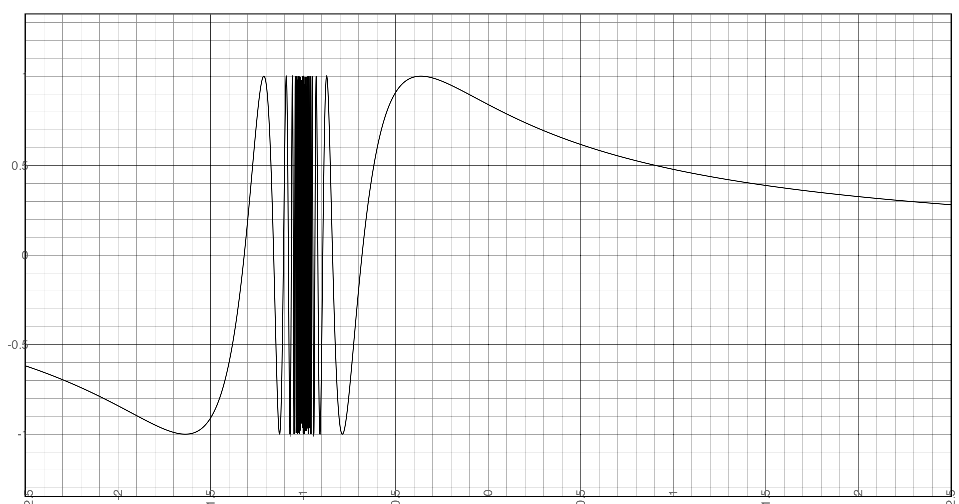

Figure 2: $a = -1$.

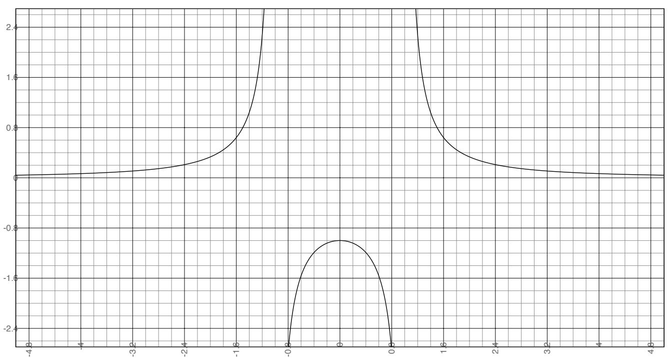

Figure 3: $a = 1$.

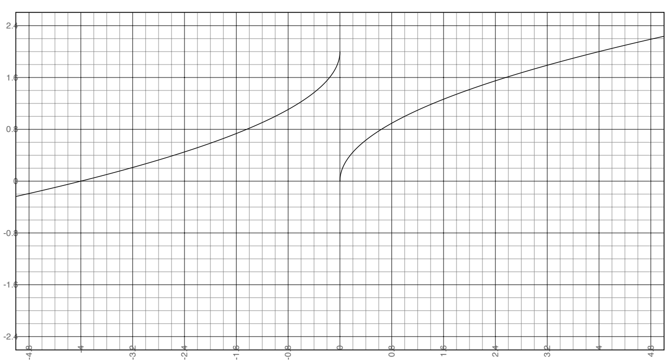

Figure 4: $a = 0$.

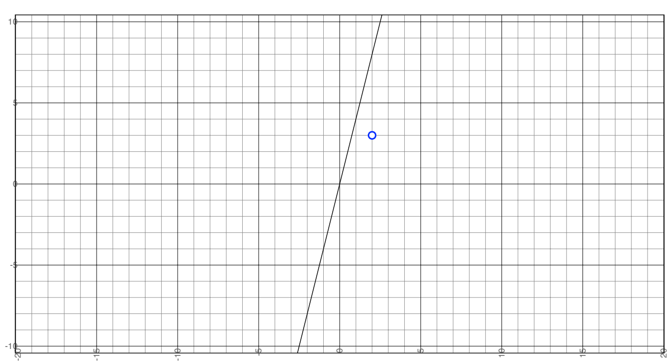

Figure 5: $a = 2$. Note: the value of this function at 2 is 3, as represented by the little circle.