Research Methods in Mathematics
Homework 6

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Due Thursday Oct 14.

(1) *Learning concepts from class.* Show that the function is continuous:
   (a) $e(x)$, where $e(x) = 0$ for $x \leq 0$ and $e(x) = x$ for $x \geq 0$, at $x = 0$.
   (b) $f(x) = x^2 + 2x$ at $x = 2$.
   (c) $f(x) = x^2 + 2x$ at $x = a$.

(2) *Developing concepts from class.* Show that $f(x) = x^{-1}$ is continuous at $x = a$ for any $a \neq 0$. The following preliminary steps may be helpful: Consider a point $x = a + h$. Show that

$$\frac{1}{a+h} - \frac{1}{a} = -\frac{h}{a^2(1+h/a)}.$$

Show that if $|h| < |a|/2$ then $\frac{1}{2} \leq |1 + h/a|$.

(3) *Developing concepts from class.* Define

$$g(x) := \begin{cases} \cos(1/x^2), & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Show that g is not continuous at 0.

(4) *Developing concepts from class.* (a) Suppose that $f_1(x) \to L_1$ as $x \to a$ and that $f_2(x) \to L_2$ as $x \to a$. Let $f = f_1 + f_2$, i.e., $f(x) = f_1(x) + f_2(x)$. Prove that $f(x) \to L_1 + L_2$ as $x \to a$.
(b) If f_1 and f_2 are continuous at a, show that f is again continuous at a.

(5) *Developing concepts from class.*
   (a) Suppose that $f(x) \to L$ as $x \to a$. Let $g = f^2$, i.e., $g(x) = f(x)^2$. Show that $g(x) \to L^2$ as $x \to a$. [Hint: Start by examining the argument used to prove that $x^2 \to a^2$ as $x \to a$.]
   (b) Show that if f is continuous at a then so is g.
   (c) Prove that if p and q are real numbers then

$$pq = \frac{1}{4}[(p+q)^2 - (p-q)^2].$$

(d) Use (b) and (c), and Q4, to prove that if $f_1(x) \to L_1$ and $f_2(x) \to L_2$ as $x \to a$ then $f_1(x)f_2(x) \to L_1 + L_2$ as $x \to a$. Deduce that the product $f_1 f_2$ of continuous functions is continuous.

(6) *Learning concepts from class.* Suppose that the domain of the continuous function f is an open interval containing $[0,1]$ and that $0 \leq f(x) \leq 1$ for each $x \in [0,1]$. Prove that there is some x such that $f(x) = x$. [Hint: apply the intermediate value theorem to $g(x) := x - f(x)$.]

(7) *Learning concepts from class.* Working carefully from the definition of a limit, prove that for any function f defined near a, $\frac{f(a+h)-f(a)}{h} \to L$ as $h \to 0$ if and only if $\frac{f(x)-f(a)}{x-a} \to L$ as $x \to a$.

(8) *Learning concepts from class.* Starting from the definition, differentiate: (a) $f(x) = Ax + B$; (b) $f(x) = (x+1)^2$; (c) $f(x) = x^4$.

(9) *Developing concepts from class.* In this question, we'll use the definition of derivative to show that if $f(x) = x^{-1}$ then $f'(x) = -x^{-2}$.

(a) Take $a \neq 0$ and $h \neq 0$. Show that

$$\frac{f(a+h) - f(a)}{h} = -\frac{1}{a^2}\frac{1}{(1+h/a)}.$$

(b) Show that

$$\frac{1}{(1+h/a)} = 1 - \frac{h/a}{(1+h/a)}$$

and deduce that

$$\frac{f(a+h) - f(a)}{h} = -\frac{1}{a^2} - \frac{h}{a^3(1+h/a)}.$$

(c) Show that

$$-\frac{h}{a^3(1+h/a)} \to 0 \quad \text{as } h \to 0.$$

(You may find question 2 helpful here.) Deduce that $f'(a) = -a^{-2}$.