Research Methods in Mathematics
Homework 7

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Due Thursday Oct 21.

Recall that if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is a 2×2 matrix, there is a matrix transformation $L_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ which sends the vector $\mathbf{v}$ to vector $A\mathbf{v}$. For the standard basis vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$L_A\mathbf{e}_1$ is the first column of A, and $L_A\mathbf{e}_2$ the second column.

(1) *Learning concepts from class.* For each matrix A, draw $A\mathbf{e}_1$ and $A\mathbf{e}_2$ on a diagram. Use this to give a geometric interpretation of L_A. (For instance, L_A might be a reflection in the x-axis, or a rotation by a certain angle about the origin.)

(a)
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c)
$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(2) *Learning concepts from class.* Write down a matrix A whose associated transformation L_A

(a) Is a dilation (scaling) by a factor of 2: $L_A\mathbf{v} = 2\mathbf{v}$.

(b) Is a rotation by 180°.

(c) Is a rotation by 90° counterclockwise.

(d) Is a reflection in the y-axis.

(e) Is a rotation by 30° clockwise.

(3) *Developing concepts from class.* The *image* of L_A is the set of all vectors which may be written as $L_A(\mathbf{v})$ for some vector $\mathbf{v}$. Suppose that A is 2×2 matrix with at least one non-zero entry. Prove that the image of L_A is a line when one column of A is a multiple of the other, and is the whole plane $\mathbb{R}^2$ otherwise.

(4) *Learning concepts from class.* Consider the following transformation rule:

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ v_2 + 1 \end{bmatrix}.$$

Describe the geometric effect of T. Is there a matrix A such that $T = L_A$? Justify your answer.

(5) *Learning concepts from class.* Let

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

What is A^2? What is A^n for a natural number n? Justify your answer by induction on n.

(6) *Learning concepts from class.* Let

$$A = \begin{bmatrix} \cos \pi/5 & -\sin \pi/5 \\ \sin \pi/5 & \cos \pi/5 \end{bmatrix}.$$

Describe L_A geometrically. Use your answer to help you calculate A^{2010}.

(7) *Developing concepts from class.*
Let

$$S = S_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

(a) Suppose $\mathbf{v} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. Show that $S\mathbf{v} = \mathbf{v}$.

(b) Find a non-zero vector $\mathbf{w}$ that makes a 90° angle with $\mathbf{v}$ and show that $S\mathbf{w} = -\mathbf{w}$.

(c) Explain why L_S is reflection in the line passing through the origin and the point $(\cos\theta, \sin\theta)$.

(d) Calculate the product $S_\alpha S_\beta$ and show that it is a rotation matrix. What is the angle of rotation?

(e) How do periscopes work? How is this relevant?

(8) *Problem.* In his recent ad for the Putnam contest practice sessions, Dave Rusin gave as a sample (essentially) the following problem: *Show that there is no equilateral triangle in the plane, each of whose vertices has rational coordinates.* Can you solve it? Spoiler alert: A hint can be found over the page.

Hint for question 8. If the coordinates of the vertices are represented by vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, work out the components of $\mathbf{b} - \mathbf{c}$ in terms of those of $\mathbf{a} - \mathbf{c}$.