Research Methods in Mathematics
Homework 8

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Due Thursday Oct 28.

In two of the questions, you will need the concept of the *matrix transpose.* If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A-transpose is

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Notice that $(A^T)^T = A$.

(1) *Learning concepts from class.* Prove the identity $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$. (Note: this should be very straightforward: I've included it because it's such a useful identity.)

(2) *Learning concepts from class.* Show algebraically that $\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

(3) *Developing concepts from class.* The law of cosines is a generalization of the Pythagorean theorem. In a triangle ABC with a right-angle at C, one has

$$|AB|^2 - |AC|^2 - |BC|^2 = 0.$$

The law of cosines says that if the angle at C is θ, one has

$$|AB|^2 - |AC|^2 - |BC|^2 = 2|AC||BC|\cos\theta.$$

(a) Suppose that the positions of A, B and C are represented by vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, so that, for example, $AC = \mathbf{c} - \mathbf{a}$. Use the law of cosines to prove that

$$(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c}) = \|\mathbf{a}-\mathbf{c}\|\|\mathbf{b}-\mathbf{c}\|\cos\theta.$$

(b) Deduce from (a) that for any vectors $\mathbf{a}$ and $\mathbf{b}$ making an angle θ, we have the formula

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta.$$

Conclude that $\mathbf{a}\cdot\mathbf{b} = 0$ if and only if the vectors meet at right-angles.

(4) *Developing concepts from class.* In this question we prove the 'dot-product formula' $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$ in a different way, not using the law of cosines.

(a) Explain why each side of the dot-product formula is unchanged if we apply the same rotation rot_ϕ to both $\mathbf{a}$ and $\mathbf{b}$ (replacing them by $rot_\phi(\mathbf{a})$ and $rot_\phi(\mathbf{b})$).

(b) I claim that it is sufficient to prove the dot-product formula in the special case where $\mathbf{a}$ points along the positive x-axis, in the sense that we can deduce the general case from this special case. Explain how this is done.

(c) Prove the dot-product formula in the special case where $\mathbf{a}$ points along the positive x-axis. Deduce that it holds in general.

(5) *Developing concepts from class.* Fix a non-zero vector $\mathbf{n}\in\mathbb{R}^2$. Let $\lambda = \{\mathbf{x}\in\mathbb{R}^2 : \mathbf{x}\cdot\mathbf{n} = 0\}$. Exhibit a vector $\mathbf{v}$ such that $\lambda = \{s\mathbf{v} : s\in\mathbb{R}\}$ (so λ is a straight line through the origin). Geometrically, how is the line λ related to $\mathbf{n}$?

(6) *Developing concepts from class.* Take a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [\mathbf{c}_1\,\mathbf{c}_2].$$

Prove that the following conditions are equivalent (that is: if one is true, all are true):

(a) $\det(A) = 0$.

(b) There are constants s and t such that $s\mathbf{c}_1 + t\mathbf{c}_2 = \mathbf{0}$.

(c) Either $\mathbf{c}_1$ is a multiple of $\mathbf{c}_2$, or vice versa.

(d) $\det A^T = 0$.

(e) There are constants s and t such that $s\mathbf{r}_1 + t\mathbf{r}_2 = \mathbf{0}$. Here $\mathbf{r}_1 = a\mathbf{e}_1 + b\mathbf{e}_2$ and $\mathbf{r}_2 = c\mathbf{e}_1 + d\mathbf{e}_2$ (so they are formed from the rows of A).

(f) Either $\mathbf{r}_1$ is a multiple of $\mathbf{r}_2$, or vice versa.

[*Hint.* Prove (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(a); then prove (d)$\Rightarrow$(e)$\Rightarrow$(f)$\Rightarrow$(d); lastly prove (a)$\Leftrightarrow$(d).]

(7) *Developing concepts from class.* Suppose that $\det A = 0$, but assume that A is not the zero-matrix.

(a) Find a non-zero vector $\mathbf{v}$ such that $A\mathbf{v}$ is a non-zero multiple of $\mathbf{v}$.

(b) Show that there exists a non-zero vector $\mathbf{w}$ such that $A\mathbf{w} = \mathbf{0}$.

(8) *Learning concepts from class.* Recall that the inverse of a 2×2 matrix A satisfies $A^{-1}A = I$. By using the formula we used to define A^{-1}, check that $AA^{-1} = I$. Show that if B is a matrix such that $BA = I$ then $B = A^{-1}$.

(9) *Developing concepts from class.* Show that the following conditions on a 2×2 matrix A are equivalent:

(a) A is orthogonal (i.e., it preserves dot products).

(b) $A^T A = I$.

(c) $A^T = A^{-1}$.

(d) $AA^T = I$.

(e) A^T is orthogonal.

[*Hint*: use the criterion for orthogonality in terms of the columns.]