Research Methods in Mathematics
Homework 9

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Due Thursday November 11.

(1) Suppose that $\star$ is a product rule for 3-vectors such that the identities $(\mathbf{a}+\mathbf{b})\star\mathbf{c} = \mathbf{a}\star\mathbf{c}+\mathbf{b}\star\mathbf{c}$ and $\mathbf{c}\star(\mathbf{a}+\mathbf{b}) = \mathbf{c}\star\mathbf{a}+\mathbf{c}\star\mathbf{b}$, as well as the identity $\mathbf{a}\star\mathbf{a} = \mathbf{0}$. Show that one always has $\mathbf{b}\star\mathbf{c} = -\mathbf{c}\star\mathbf{b}$.

(2) Prove the following identity for 3-vectors:
$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})+\mathbf{v}\times(\mathbf{w}\times\mathbf{u})+\mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0}.$$

(3) Prove the identity
$$\|\mathbf{a}\times\mathbf{b}\|^2+(\mathbf{a}\cdot\mathbf{b})^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2.$$
Use this and the cosine formula for the dot product to deduce the formula
$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$$
where θ is the angle from $\mathbf{a}$ to $\mathbf{b}$ (and $0\leq\theta\leq\pi$). Give geometric conditions equivalent to the algebraic condition
$$\mathbf{a}\times\mathbf{b} = \mathbf{0},$$
justifying your answer.

(4) Prove algebraically that if $\mathbf{a}\in\mathbb{R}^3$ is any non-zero vector, one can find non-zero vectors $\mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c} = 0$.

(5) Recall from class that $S_{\mathbf{c}}$ denotes the matrix representing reflection in the plane orthogonal to $\mathbf{c}$. Let $P = [\mathbf{abc}]$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are mutually orthogonal vectors, each of length 1. Let $Q = PS_{\mathbf{e}_3}P^T$. Show that $Q = S_{\mathbf{c}}$.

(6) Let
$$X = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 9 \\ 0 & 0 & -2 \end{bmatrix}$$
Find the characteristic polynomial $\chi_X(t)$. Find the eigenvalues of X (they are just the roots of the characteristic polynomial). For each eigenvalue λ, find an eigenvector, i.e., a non-zero vector $\mathbf{v}$ such that $X\mathbf{v} = \lambda v$. (You will need to solve the system of equations $(X-\lambda I)\mathbf{v} = \mathbf{0}$.)

(7) Repeat the previous question for

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

For one of the eigenvalues λ, you should be able to find *two* eigenvectors, neither a multiple of the other.