

## Research Methods in Mathematics, Lecture 10

### Examples of limits

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In the last lecture we gave the definition of a limit. Here's a reminder: if  $f$  is a function whose domain includes all real numbers close to  $a$ , except perhaps  $a$  itself, we say that  $f(x) \rightarrow L$  if for every  $\epsilon > 0$  there's a  $\delta > 0$  (which may depend on  $\epsilon$ ) such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Let's now work through some examples. Before we begin, here's a useful protocol to follow when proving that some function limits to a (known) number  $L$  as  $x \rightarrow a$ .

- *Exploration stage.* Begin by writing down the expression  $|f(x) - L|$  using the definitions of  $f$  and  $L$ . Simplify it as much as you can.
- Now try to show that  $|f(x) - L|$  expression is  $\leq$  some expression involving  $|x - a|$ . Remember, you can use  $=$ ,  $\leq$  and  $<$ , but it's pointless to bring in any  $>$  or  $\geq$  signs.
- If you've shown that  $|f(x) - L| \leq$  (some function of  $|x - a|$ ), substitute in the inequality  $|x - a| < \delta$ , so as to have  $|f(x) - L| <$  (some function of  $\delta$ ) (provided that  $0 < |x - a| < \delta$ ). We want this function of  $\delta$  to be equal to  $\epsilon$ . So, given  $\epsilon$ , solve for  $\delta$ .
- *Formal proof stage.* Now we're ready to write out the proof. This should start like this: "Given  $\epsilon > 0$ , take  $\delta = \cdot$ ". You put in the value of  $\delta$  that you worked out in the exploration stage; or if you're still not sure, leave it blank for now and fill it in later. Continue: "Then, when  $0 < |x - a| < \delta$ , we have  $|f(x) - L| = \dots$ ". Write out the algebra and the inequalities you worked out, including the substitution  $|x - a| < \delta$ . Finally plug in the chosen value of  $\delta$  and show that it equals  $\epsilon$ . You're done!

**Example 1** Let  $f(x) = 3x - 2$ . Then  $f(x) \rightarrow 3a - 2$  as  $x \rightarrow a$ .

Exploration: we have

$$|f(x) - (3a - 2)| = |(3x - 2) - (3a - 2)| = |3x - 3a| = 3|x - a|.$$

The expression on the right is a function of  $|x - a|$ . Say  $|x - a| < \delta$ . Then we have

$$|f(x) - (3a - 2)| = 3|x - a| < 3\delta.$$

Given  $\epsilon$ , let's choose  $\delta$  so that  $3\delta = \epsilon$ , i.e., choose  $\delta = \epsilon/3$ .

Formal proof: Given  $\epsilon > 0$ , take  $\delta = \epsilon/3$ . Then, if  $0 < |x - a| < \delta$ , we have

$$|f(x) - (3a - 2)| = 3|x - a| < 3\delta = 3\epsilon/3 = \epsilon.$$

Hence  $f(x) \rightarrow 3a - 2$  as  $x \rightarrow a$ .

**Example 2** Let  $f(x) = x^2$ . Then  $f(x) \rightarrow a^2$  as  $x \rightarrow a$ .

This one is quite a lot harder. *Exploration.* We have

$$|f(x) - a^2| = |x^2 - a^2| = |x + a||x - a|.$$

The term  $|x - a|$  is good, but  $|x + a|$  is troublesome. But we can relate it to  $|x - a|$ :

$$|x + a| = |(x - a) + 2a| \leq |x - a| + |2a| = |x - a| + 2|a|.$$

So

$$|x^2 - a^2| = |x + a||x - a| \leq (|x - a| + 2|a|)|x - a|.$$

So, if  $|x - a| < \delta$ , we have

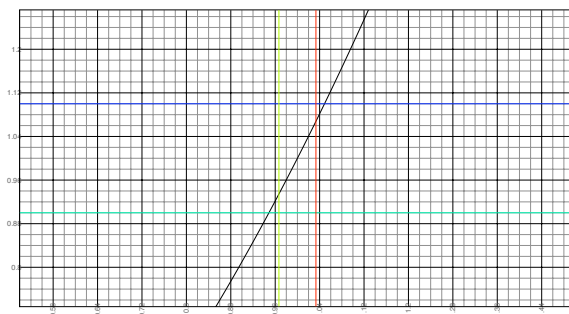
$$|x^2 - a^2| < (\delta + 2|a|)\delta = \delta^2 + 2|a|\delta.$$

At this point, we could let this function of  $\delta$  equal  $\epsilon$ , and use the quadratic formula to solve for  $\delta$ . Maybe neater is to observe that if  $\delta$  is small then  $\delta^2$  is *really* small, so the  $\delta^2$  term oughtn't to matter much. Specifically, if we choose  $\delta \leq 1$  then  $|x^2 - a^2| < \delta^2 + 2|a|\delta \leq \delta + 2|a|\delta = (1 + 2|a|)\delta$ . This suggests that we should take  $\delta$  to be either 1 or  $(1 + 2|a|)^{-1}\epsilon$ , whichever is smaller.

*Formal proof.* Given  $\epsilon > 0$ , take  $\delta = \min\{1, (1 + 2|a|)^{-1}\epsilon\}$ . Then, if  $0 < |x - a| < \delta$ , we have

$$\begin{aligned} |f(x) - a^2| &= |x^2 - a^2| \\ &= |x + a||x - a| \\ &\leq (|x - a| + 2|a|)|x - a| \\ &< (\delta + 2|a|)\delta && \text{since } |x - a| < \delta \\ &\leq (1 + 2|a|)\delta && \text{since } \delta \leq 1 \\ &\leq \frac{1 + 2|a|}{1 + 2|a|}\epsilon = \epsilon. \end{aligned}$$

So  $f(x) \rightarrow a^2$  as  $x \rightarrow a$ .



Let's relate this example to our graphical interpretation of limits. Let's say we take  $a = 1$  and  $\epsilon = 1/10$ . Then our proof says that we should take  $\delta = 1/30$ . We can see that this really works in the following picture of the graph of  $x^2$  (it looks almost straight because we've zoomed in). The horizontal lines are at  $y = 1 + \epsilon$  and  $y = 1 - \epsilon$ , while the vertical lines are at  $x = 1 - \delta$  and  $x = 1 + \delta$ . As you can see, when  $x$  lies between these two values, the graph of  $f$  stays inside the box.

**Example 3** Let  $f(x) = x \sin x$ . Then  $f(x) \rightarrow 0$  as  $x \rightarrow 0$ .

*Exploration.* We have  $|f(x) - 0| = |x \sin x| = |\sin x| \cdot |x|$ . The term  $|\sin x|$  is less troublesome than it looks: we have  $|\sin x| \leq 1$  for any  $x$ , so  $|f(x)| \leq 1 \cdot |x|$ . So, if  $0 < |x| < \delta$ , we have  $|f(x)| < \delta$ . So given  $\epsilon$ , we can take  $\delta = \epsilon$ .

*Formal proof.* Given  $\epsilon > 0$ , let  $\delta = \epsilon$ . Then, when  $0 < |x| < \delta$ , we have

$$\begin{aligned} |f(x)| &= |\sin x| |x| \\ &\leq |x| && \text{since } |\sin x| \leq 1 \\ &< \delta = \epsilon. \end{aligned}$$

So  $f(x) \rightarrow 0$  as  $x \rightarrow 0$ .

**Example 4** Let  $f(x) = 1$  for  $x \neq 0$  and define  $f(0) = 23$ . Then  $f(x) \rightarrow 1$  as  $x \rightarrow 0$ .

*Exploration.* If  $0 < |x| < \delta$  then  $x \neq 0$ , so  $f(x) = 1$ . Hence  $|f(x) - 1| = 0$ . So any  $\delta$  will do!

*Formal proof.* Given  $\epsilon > 0$ , take  $\delta = 463$ . Then, when  $0 < |x| < \delta$ , we have  $x \neq 0$  and hence

$$|f(x) - 1| = 0 < \epsilon.$$

Hence  $f(x) \rightarrow 1$  as  $x \rightarrow 0$ .

This last example is very straightforward, but it makes the point that the value of  $f(a)$  (if it exists at all) has no bearing on the value of the limit of  $f(x)$  as  $x \rightarrow a$ .