Research Methods in Mathematics, Lecture 11
Continuity and the intermediate value theorem

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Continuity

Consider the following examples of limits.

Example 1 Let $f(x) = x\sin(1/x)$ for $x \neq 0$. Then 0 is not in the domain in f. Nonetheless, $f(x) \to 0$ as $x \to 0$. (Indeed, given $\epsilon < 0$, let $\delta = \epsilon$. Then, whenever $0 < |x| < \delta$, we have

$$|f(x) - 0| = |x||\sin(1/x)| \leq |x| < \delta = \epsilon,$$

which proves that $f(x) \to 0$ as $x \to 0$. Here we used that $|\sin(1/x)| \leq 1$ for any x.)

Example 2 Let $f(x) = x\sin(1/x)$ for $x \neq 0$, and define $f(0) = 0$, so that the domain is now $\mathbb{R}$. It's still the case that $f(x) \to 0$ as $x \to 0$ (why?). So we can say in this case that $f(x) \to f(0)$ as $x \to 0$.

Example 3 Let $f(x) = x\sin(1/x)$ for $x \neq 0$, and define $f(0) = 19$. The domain is $\mathbb{R}$. It's still the case that $f(x) \to 0$ as $x \to 0$. But here, it is not true that that $f(x) \to f(0)$ as $x \to 0$.

The difference between the second and third examples is captured by the notion of continuity:

Definition 4 Suppose that the domain of the function f includes a and all points close to a. We say that f is continuous at a if $f(x) \to a$ as $x \to a$. If the domain includes an interval $I = (x_0, x_1)$, we say that f is continuous on I if it is continuous at each $a \in I$.

So, the second function above is continuous at 0, but the third is not.

Useful facts. The sum or product of continuous functions is continuous. A constant multiple of a continuous function is continuous.

For example, since $f(x) = x$ is continuous, and $g(x) = 1$ (constant fucntion) is continuous, every polynomial function $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is continuous.

The intermediate value theorem

The most important theorem about continuous functions is the following.

Theorem 5 (Intermediate value theorem) *Suppose that* f *is a continuous function on an open interval* (A, B) *containing the closed interval* $[a, b]$*. Suppose that* $f(a) < 0$ *and* $f(b) > 0$*. Then there is some* x *in* (a, b) *with* $f(x) = 0$*.*

This statement shows that our definition of the real numbers is on the right track. If we work not with real numbers but with the rational numbers $\mathbb{Q}$, the intermediate value theorem fails:

Example 6 *Suppose we have a function whose domain is some set of rational numbers. We say that* $f(x) \to L$ *as* $x \to x_0$ *(in the rational sense) if for every rational* $\epsilon > 0$*, there's a rational* $\delta > 0$ *such that, for all rational* x *such that* $0 < |x - x_0| < \delta$*, we have* $|f(x) - L| < \epsilon$*. We then say that* f *is continuous in the rational sense on* (a, b) *if* $f(x) \to f(x_0)$ *as* $x \to x_0$*, this for all* $x_0 \in (a, b)$*. Define a function of rational numbers* f *by setting* $f(x) = -1$ *if* $x^2 < 2$*, and* $f(x) = 1$ *if* $x^2 > 2$*. Our function is continuous in the rational sense (it would be discontinuous at a rational* x_0 *such that* $x_0^2 = 2$*. but there's no such rational!). Yet it gets from* -1 *to* 1 *without ever taking the value* 0*.*

As this example suggests, the intermediate value theorem isn't so much a theorem about functions as a theorem about the real numbers. As such, it's proof will make use of completeness.

Proof of IVT Let A be the set of all in $[a, b]$ such that $f(t) \leq 0$. Then A is non-empty (because $a \in A$). It is bounded above (by b). So, by the completeness axiom, A has a supremum $x = \sup A$.

Let $y = f(x)$. I claim that $y = 0$.

To prove the claim, we must rule out two possibilities: (i) that $y < 0$, and (ii) that $y > 0$. Suppose first that $y < 0$. Let $\epsilon = -y/2$. Then there exists $\delta > 0$ such that, when $0 < |x' - x| < \delta$, we have $|f(x') - f(x)| < \epsilon$. Let $x' = x + \delta/2$. Then $0 < |x' - x| < \delta$, so $|f(x') - y| < \epsilon$, and hence $f(x') < y + \epsilon = y/2 < 0$. But this says that $x' \in A$, contradicting the fact that it is bigger than the supposed upper bound x.

Now suppose that $y > 0$. Let $\epsilon = y/2$. Then there exists $\delta > 0$ such that, when $0 < |x' - x| < \delta$, we have $|f(x') - f(x)| < \epsilon$. Then for any $x' \in [x - \delta, x]$, we will have $|f(x') - y| < \epsilon$, and hence $f(x') > y - \epsilon = y/2 > 0$. Hence $x - \delta/2$ is an upper bound for A, contradicting the fact that x is the least upper bound.

This leaves only the possibility that $y = 0$, and hence finishes the proof. $\square$

Corollary 7 *Let p be a polynomial of odd degree and leading coefficient 1, say*

$$p(x) = x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0.$$

Then there exists a real number x with $p(x) = 0$.

Proof I claim that for x large enough, $p(x) > 0$. Indeed, suppose $x > 1$. Then

$$\begin{aligned} |a_{2n}x^{2n} + \cdots + a_1 x + a_0| &\le |a_{2n}|x^{2n} + |a_{2n-1}|x^{2n-1} + \cdots + |a_0| \\ &\le |a_{2n}|x^{2n} + |a_{2n-1}|x^{2n} + \cdots + |a_0|x^{2n} \\ &= (|a_{2n}| + |a_{2n-1}| + \cdots + |a_0|)x^{2n}. \end{aligned}$$

Put $C = \max\{1, |a_{2n}| + |a_{2n-1}| + \cdots + |a_0|\}$. Then, when $x > C$,

$$\begin{aligned} p(x) &= x^{2n+1} + (a_{2n}x^{2n} + \cdots + a_1 x + a_0) \\ &\ge x^{2n+1} - (|a_{2n}|x^{2n} + |a_{2n-1}|x^{2n-1} + \cdots + |a_0|) \\ &\ge x^{2n+1} - (|a_{2n}| + |a_{2n-1}| + \cdots + |a_0|)x^{2n} \\ &\ge x^{2n+1} - Cx^{2n}. \end{aligned}$$

But since $x > C$, we have

$$x^{2n+1} - Cx^{2n} > x^{2n+1} - x^{2n+1} = 0.$$

So $p(x) > 0$.

Similarly (try it!!), when $x < -C$, we have $p(x) < 0$.

We pointed out earlier that polynomial functions are continuous. So, by the IVT, p has a zero somewhere in $[-C, C]$. □

Spivak reference: Chapter 6, Chapters 7–8 (only one of the 'hard theorems').