Research Methods in Mathematics
Lecture 12: Differentiation

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Derivatives

Now that we have a valid notion of limit, we can give a definition of the derivative of a function.

The idea of the derivative is that it should capture the instantaneous rate of change of a function: if the function is 'distance traveled in time t', the derivative at $t = t_0$ is what the speedometer reads at time t_0.

Let f be a function defined on some interval (c,d), and take a point $a \in (c,d)$. Let h be a small but non-zero real number. It's easy to measure the average rate of change of f over the interval $[a, a+h]$ (or $[a+h, a]$, if h happens to be negative): this is the ratio

$$\frac{f(a+h) - f(a)}{h}$$

of the change in f to the change in x. The instantaneous rate of change, or derivative, is what we get by letting h become very small. To be precise:

Definition 1 Let f be a function defined in an interval (c,d), and suppose $a \in (c,d)$. We say that f is *differentiable at a* if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If so, we denote this limit by $f'(a)$ and call it the derivative of f at a. If the domain of f is an interval (x,y), and f is differentiable at every point in (x,y), we say that f is a differentiable function and denote by f' the derivative function $a \mapsto f'(a)$.

Note that the limit can also be written as

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a};$$

to relate the two, take $x = a + h$.

Example 2 Let $f(x)$ be a constant function, $f(x) = c$. Then $f'(a) = 0$ for any a. Indeed,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Example 3 Let $f(x) = cx + d$. Then $f'(a) = c$ for any a. Indeed,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{c(a+h)-ca}{h} = \lim_{h\to 0} c = c.$$

Example 4 Let $f(x) = x^3$. Then $f'(x) = 3x^3$. To prove this, note that $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$, so $f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3$. Hence

$$f'(x) = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} 3x^2 + 3xh + h = 3x^2.$$

There's a great deal to say about derivatives, and we have time for very little of it. One point is that differentiable functions have the following property (which in fact completely characterizes them): *if you zoom in on the graph of a differentiable function, focusing your microscope at the point $(a, f(a))$, then if you turn the magnification high enough, the graph will be as close as you like to a straight line of slope $f'(a)$.*

This is illustrated by the zoom sequence over the page.

Since a differentiable function looks linear when viewed at high magnification, it ought to be continuous. We'll finish by proving that this is so.

Theorem 5 *If f is differentiable at a then it's continuous at a.*

Proof Since it's differentiable, the ratio $[f(a+h)-f(a)]/h$ approaches a limit $f'(a)$ as $h \to 0$. That means: given any $\epsilon > 0$, there exists a number $\eta > 0$ (I'm going to reserve δ for later use!) such that whenever $0 < |h| < \eta$, we have

$$\left| \frac{f(a+h)-f(a)}{h} - f'(a) \right| < \epsilon.$$

For continuity, we need to show that $|f(a+h) - f(a)|$ is small. Specifically, we have to show that $f(a+h) \to f(a)$ as $h \to 0$. But observe that when $0 < |h| < \eta$,

$$\big| f(a+h) - f(a) - f'(a)|h| \big| < \epsilon |h|$$

and hence

$$\begin{aligned} |f(a+h) - f(a)| &= \big| f(a+h) - f(a) - f'(a)|h| + f'(a)|h| \big| \\ &\leq \big| f(a+h) - f(a) - f'(a)|h| \big| + |f'(a)||h| \\ &< (\epsilon + |f'(a)|)|h| \\ &\leq (1 + |f'(a)|)|h| \qquad \text{provided that } \epsilon \leq 1 \\ &< (1 + |f'(a)|)\epsilon. \end{aligned}$$


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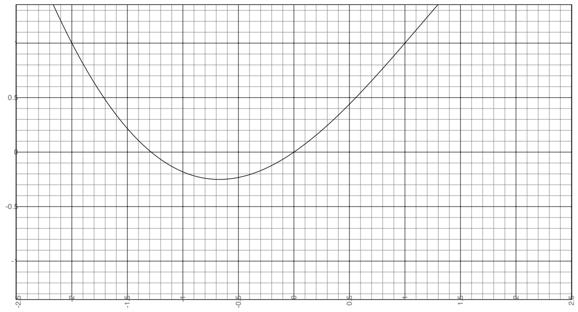
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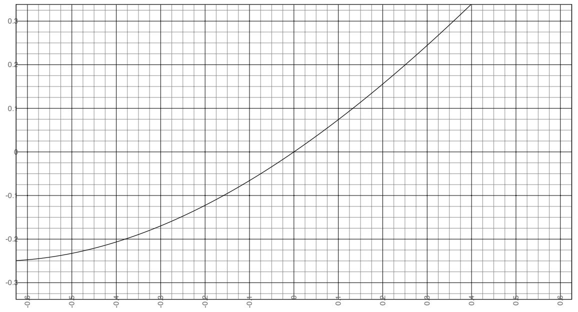
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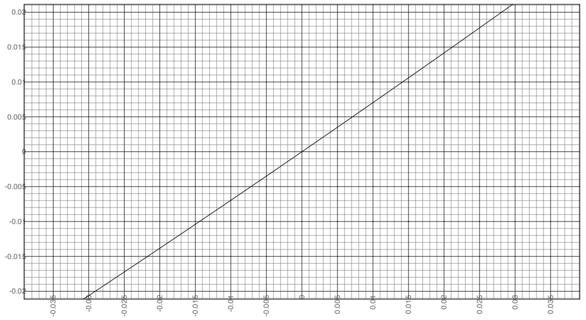
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Given $\epsilon > 0$, let $\epsilon' = \min\{1, \epsilon\}$. We'd like to make $(1 + |f'(a)|)\eta$ equal to ϵ' (and so $\leq \epsilon$). So, let η be as above, and let

$$\delta = \min\{(1 + |f'(a)|)^{-1}\epsilon', \eta\}.$$

Then, when $0 < |x - a| < \delta$, it's also true that $0 < |x - a| < \eta$, so by our calculation,

$$|f(a + h) - f(a)| < (1 + |f'(a)|)\eta \leq \epsilon' \leq \epsilon,$$

which shows that $f(a + h) \to f(a)$ as $h \to 0$. $\square$

Spivak reference: Chapter 9.