Freshman Research Initiative:
Research Methods in Mathematics

T. PERUTZ

2 Counting and induction

2.1 Number systems

> In science nothing capable of proof ought to be accepted without proof. Though this demand seems entirely reasonable, I cannot regard it as having been met even in the most recent methods of laying the foundations of the simplest science; viz., that part of logic which deals with the theory of numbers. In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought. [...] It is only through the purely logical process of building up the science of numbers that we are prepared to investigate our notions of space and time by bringing them into relation with this number-domain created in our mind.
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> RICHARD DEDEKIND, *The nature and meaning of numbers*, 1887

Dedekind's belief that the number-concept is entirely independent of the notions or intuitions of space and time was shared by some but not all of his contemporaries. Today we have a much more detailed understanding of the subtleties inherent in this issue—what is capable of proof and what is not—but the issue has still not been fully resolved. What we can say unequivocally is this:

- We can make precise how the system of natural numbers, $1, 2, 3, \dots, 1\,000\,000, \dots$ *works*, provided that we take for granted that such a system *exists.* (We have to start *somewhere*.)
- Starting from the natural numbers, we can *define* all the other number systems of interest, including the integers, the rational numbers, the real numbers and the complex numbers.

In this lecture, we will look at the first of these two ideas.

2.1.1 Counting

The 'natural numbers' are the counting numbers 1, 2, 3, and so on.[1] We shall write $\mathbb{N}$ for the natural numbers collectively.

But what are the natural numbers?

Maybe a better question is, how do we count things? We keep track of our running total. If there is something to count at all, then when we count it we have 1. If there is something else to count, we add one to it. The most basic concepts, from this point of view, are '1' and 'adding one'. This idea leads to the following axioms for natural numbers:

AXIOMS:

- There is a natural number 1.
- For any natural number n, there is a natural number $S(n)$, its 'successor' (think: $n+1$). So we have natural numbers 1, $S(1)$, $S(S(1))$, $S(S(S(1)))$, and so on (think: $1,2,3,4$).
- There is no natural number n such that $S(n) = 1$ (think: 0 is not a natural number).
- If $S(n) = S(m)$ then $n = m$.

There is one other statement connected with counting which is more subtle, but very important. We propose it as an axiom too.

PRINCIPLE OF MATHEMATICAL INDUCTION

- Let $P(n)$ be a statement which takes as input a natural number n. Suppose that $P(1)$ is true. Suppose that, whenever $P(m)$ is true, $P(S(m))$ is also true. Then $P(n)$ is true for all natural numbers n.

These axioms chime with our intuitive understanding of natural numbers, and so they seem to be a reasonable place to start. The principle of induction deserves further comment. It is a 'domino' principle: if one domino knocks over the next, then if you knock over the first domino they all eventually fall. Note that you have to prove two things: the *base step* $P(1)$, and the *inductive step*, that $P(n)$ implies $P(n+1)$.

Example 2.1 Suppose $P(n)$ is the following statement:

$$P(n)\colon\ n(n+1)(2n+1) \text{ is divisible by 6.}$$

[1] There is some disagreement about whether zero is a natural number or not. For us, it is not.

This statement invokes two concepts that we have not yet discussed: addition and multiplication of natural numbers. For now, we will take them for granted. We will prove $P(n)$ by induction.

Well, $P(1)$ is the statement that $1.2.3$ is divisible by 6, which is true as $1.2.3 = 6$. Supposing $P(n)$ to be true, we prove $P(n+1)$. Note that $P(n+1)$ is the statement that

$$(n+1)(n+2)(2(n+1)+1) \text{ is divisible by } 6.$$

Well,

$$\begin{aligned}(n+1)(n+2)(2(n+1)+1) &= (n+1)(n+2)((2n+1)+2)\\ &= (n+1)n(2n+1) + (n+1)(2)(2n+1) + (n+1)(n+2)(2)\\ &= n(n+1)(2n+1) + 2(n+1)(2n+1+n+2)\\ &= n(n+1)(2n+1) + 2(n+1)(3n+3)\\ &= n(n+1)(2n+1) + 6(n+1)^2.\end{aligned}$$

The term $n(n+1)(2n+1)$ is divisible by 6 by our inductive assumption, and $6(n+1)^2$ is divisible by 6, so $(n+1)(n+2)(2(n+1)+1)$ is divisible by 6, which is $P(n+1)$. We conclude that $P(n)$ is true for all natural numbers n.

This example illustrates what is both a strength and a weakness of the inductive method: the proof gives no clue to the inspiration behind the statement! This is a weakness because the statement may still seem mysterious, but a strength because, if someone gives us an assertion, we may be able to prove it even though we don't understand 'why' it is true.