Research Methods in Mathematics (FRI)
Fall Semester, 2010
Part I: From counting to calculus
Lecture 2: Counting and induction

Tim Perutz

August 31, 2010

Number systems

From R. Dedekind, *The nature and meaning of numbers*, 1887

> *In science nothing capable of proof ought to be accepted without proof. Though this demand seems entirely reasonable, I cannot regard it as having been met even in the most recent methods of laying the foundations of the simplest science; viz., that part of logic which deals with the theory of numbers. In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought.*

Number systems

Dedekind's belief that the number-concept is entirely independent of the notions or intuitions of space and time was shared by some but not all of his contemporaries. Today we have a much more detailed understanding of the subtleties inherent in this issue—what is capable of proof and what is not—but the issue has still not been fully resolved.

Number systems

What we can say unequivocally is this:

- We can make precise how the system of natural numbers, $1, 2, 3, \ldots, 1,000,000, \ldots$ *works*, provided that we take for granted that such a system *exists.* (We have to start somewhere!)
- Starting from the natural numbers, we can *define* all the other number systems of interest, including the integers, the rational numbers, the real numbers and the complex numbers.

In this lecture, we will look at the first of these two ideas.

Turtles

Why take for granted that a system of natural numbers exists? Well, we have to start somewhere! Otherwise we'd either have a circularity, or it would be turtles all the way down.

> *A well-known scientist once gave a public lecture on astronomy. He described how the earth orbits around the sun and how the sun, in turn, orbits around the center of a vast collection of stars called our galaxy. At the end of the lecture, a little old lady at the back of the room got up and said: "What you have told us is rubbish. The world is really a flat plate supported on the back of a giant tortoise." The scientist gave a superior smile before replying, "What is the tortoise standing on?"*
>
> *"You're very clever, young man, very clever," said the old lady. "But it's turtles all the way down!"*
>
> Stephen Hawking, *A Brief History of Time.*