Research Methods in Mathematics
Lecture 4: Multiplication; fractions

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Multiplication

In the last lecture, we gave an inductive definition of addition. Now we come to multiplication. Again, we define mn by induction on n: we set $m \times 1 = m$ and $m \times S(n) = m \times n + n$.

For instance,

$$n \times 3 = n \times S(2) = n \times 2 + n = n \times S(1) + n = (n + n) + n.$$

Theorem 1 *For all natural numbers m, n and p, we have*

(1) $(mn)p = m(np)$;

(2) $mn = nm$.

As an example, let's check that $n \times 2 = 2 \times n$. As usual, we'll prove this by induction. For $n = 1$, we have $2 \times 1 = 2$ and $1 \times 2 = 1 + 1 = 2$, so this checks. For the inductive step, assuming $n \times 2 = 2 \times n$ we must prove that $(n + 1) \times 2 = 2 \times (n + 1)$. We have

$$\begin{aligned}
(n + 1) \times 2 &= (n + 1) + (n + 1) && \text{plug in the definition} \\
&= n + ((1 + (n + 1)) && \text{associativity of addition} \\
&= n + (1 + (1 + n)) && \text{commutativity of addition} \\
&= n + ((1 + 1) + n) && \text{associativity again} \\
&= n + (2 + n) && \\
&= n + (n + 2) && \text{commutativity again} \\
&= (n + n) + 2 && \text{associativity again} \\
&= (n \times 2) + 2 && \text{definition of mult. by 2.}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
2 \times (n + 1) &= 2 \times S(n) \\
&= 2 \times n + 2,
\end{aligned}$$

by definition. The inductive hypothesis says that $n \times 2 = 2 \times n$. So

$$(n+1) \times 2 = n \times 2 + 2 = 2 \times n + 2 = 2 \times (n+1),$$

which proves the inductive step.

There is also a result that relates addition to multiplication:

Theorem 2 *For all natural numbers m, n and p, we have $(m+n)p = mp + np$.*

The properties of addition and multiplication described by the three theorems are fundamental ones that will be shared by several number systems which appear in this course.

Fractions

It's hard to give a definition of fractions that isn't circular. A fraction ought to be a ratio

$$\frac{p}{q}$$

where p and q are natural numbers. But what is a ratio? Isn't it the same as a fraction?

The important thing about fractions (or ratios) is that a cancelation law should hold:

$$\frac{np}{nq} = \frac{p}{q}$$

for all natural numbers n. For instance, $3/2 = 6/4 = 9/6 = 12/8 = \dots$. It's possible to turn this into a definition:

Definition 3 A fraction is represented by an ordered pair (p, q) of natural numbers. The fraction corresponding to (p, q) is denoted by p/q. For any natural number n, the fraction $(np)/(nq)$ is considered to represent the same fraction as p/q.

To make this a useful definition, we must declare the rules for fraction arithmetic. These are as follows: we define addition by

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}$$

and multiplication by

$$\frac{p}{q}\frac{r}{s} = \frac{pr}{qs}.$$

Because a fraction is not represented in a unique way by a pair of natural numbers, we must check that these definitions are self-consistent. The point is that p/q doesn't determine p and q (because we can multiply each of them by a common factor). Yet our definitions of addition an multiplication invoke p and q. To check that the definition of multiplication is valid, observe that

$$\frac{mp}{mq}\frac{nr}{ns} = \frac{(mp)(nr)}{(mq)(ns)} = \frac{(mn)(pr)}{(mn)(qs)},$$

where the second equality uses the associative and commutative properties of multiplication. Canceling, we see that

$$\frac{mp}{mq}\frac{nr}{ns} = \frac{p}{q}\frac{r}{s}$$

which shows that all is well: our definition is self-consistent. Checking the self-consistency of the definition of addition is left as an exercise.

As an example of an *invalid* definition, consider the following attempt to define an operation $\star$:

$$\frac{p}{q} \star \frac{r}{s} = \frac{p+r}{q+s}.$$

This doesn't make any sense, because, for instance,

$$\frac{2}{1} \star \frac{1}{2} = \frac{3}{3} = 1,$$

while

$$\frac{4}{2} \star \frac{1}{2} = \frac{5}{4} \neq 1,$$

yet $\frac{2}{1} = \frac{4}{2}$.