Research Methods in Mathematics
Lecture 5: Inequalities

T. PERUTZ

Comparing natural numbers

If m and n are natural numbers, we write $m > n$ (or $n < m$) to mean that there is some natural number p such that $m = n + p$.

Theorem 1 *Let m and n be any two natural numbers. Then exactly one of the following three possibilities holds:*

(1) $m = n$;

(2) $m < n$;

(3) $m > n$.

This theorem really contains two separate statements. One says that at least one of the three possibilities holds. The other says that at most one of them holds; they are mutually exclusive. We shall prove that at least one of them holds, but omit the proof of exclusivity. Here goes:

Proof Given m and n, we want to prove that (1) $m = n$, or (2) $m < n$, or (3) $m > n$. Since this statement involves two natural numbers, we have to be careful about how to apply induction. So we prove

$P(n)$: given any natural number m, either (1) $m = n$, or (2) $m < n$, or (3), $m > n$.

The base step $P(1)$ says that given any m, either $m = 1$ or $m < 1$ or $m > 1$. Note, by the way that $m < 1$ is certainly false (since 1 is not a successor). So we want to prove (for all m) the statement $Q(m)$: either $m = 1$ or $m > 1$. This we can do by another induction. We'll indent this to emphasize the 'nested' nature of this proof.

> $Q(1)$ is clearly true (as $1 = 1$), and if $Q(m)$ is true then either $m = 1$, in which case $S(m) = 1 + 1 > 1$, or else $m = 1 + p$ for some p, in which case $S(m) = S(1 + p) = 1 + S(p)$ so $S(m) > 1$. Either way, $S(m) > 1$, so $Q(S(m))$ is true. This completes the inductive proof of $Q(m)$, and hence of $P(1)$.

Now let's show that if $P(n)$ holds, so does $P(S(n))$. So we know that for any m, either (1) $m = n$ or (2) $m < n$ or (3) $m > n$. We must show that for any m, either $m = S(n)$ or $m < S(n)$ or $m > S(n)$. We'll consider the three possibilities in turn:

(1) $m = n$. In this case, $S(n) = S(m) = m + 1$, so $S(n) > m$.

(2) $m < n$. This means that $n = m + p$ for some p. But then $S(n) = S(m+p) = m + S(p)$, so $S(n) > m$.

(3) $m > n$. This means that $m = n + q$ for some q. We know that either $q = 1$ or $q > 1$ (this was the statement $Q(q)$ proved already). If $q = 1$ then $m = n + 1 = S(n)$, i.e., $S(n) = m$. If $q > 1$ then we can write $q = 1 + r$ for some r. But then $m = n + q = n + (1 + r) = (n + 1) + r = S(n) + r$, so $m > S(n)$.

In every case, we find that either $S(n) = m$ or $S(n) > m$ or $S(n) < m$, which is $P(n+1)$.

□

The integers

Whilst it is always possible to add two natural numbers, it is not always possible to subtract. For this, we need to introduce the system of integers, in which ne has 0 and negative numbers.

There are several ways to give a formal definition. Here's one.

Definition 2 Introduce a formal symbol 0. Also take two copies of the system of natural numbers, labeled 'positive' and 'negative'. An integer is either zero, or it is a 'positive' natural number n, or it is a 'negative' natural number, written $-n$.

If you're worried about the idea of taking two copies, here's one way to make it seem more reasonable. Consider pairs (n, m) of natural numbers, where we insist that m is either 1 or 2. The pairs $(n, 1)$ constitute a copy of the system of natural numbers which we label as 'positive'. The pairs $(n, 2)$ constitute another copy that we label as 'negative'. We don't want to actually write $(n, 2)$, so instead we write $-n$.

This is a very simple-minded definition. It shifts the work onto defining addition. This goes as follows:

Definition 3 Let a and b be natural numbers. We define addition of integers as follows:

- $a+0=a$ and $0+a=a$
- $(-a)+0=-a$ and $0+(-a)=-a$.
- $a+b$ is defined as for natural numbers.
- $(-a)+(-b)=-(a+b)$.
- If $a>b$ (say $a=b+c$) then $a+(-b)=c$ and $(-a)+b=-c$.
- If $a<b$ (say $b=a+d$) then $a+(-b)=-d$ and $(-a)+b=d$.
- If $a=b$ then $a+(-b)=0$.

To see that this is a complete definition, we have to invoke our theorem, which tells us that whatever a and b may be, we have $a=b$ or $a>b$ or $a<b$.