Research Methods in Mathematics
Lecture 6: Rational numbers; absolute values

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The story so far

- We began with natural numbers. We characterized the natural numbers by the axioms that there's a special natural number 1, the successor function S, that 1 is not a successor, that different natural numbers have different successors, and that a statement $P(n)$ about natural numbers n that is true for 1 and satisfies $P(n) \Rightarrow P(S(n))$ is true for all natural numbers.
- We defined addition, and noted the commutative law $m + n = m + n$ and the associative law $m + (n + p) = (m + n) + p$.
- We used addition to define inequality $m > n$. We proved that for any two natural numbers m and n, we have $m = n$, $m > n$ or $m < n$, and I asserted that at most one of these things is true.
- In a different direction, we used addition to define multiplication. We noted that multiplication is commutative ($mn = nm$) and associative ($m(np) = (mn)p$ and distributes over addition $m(n + p) = mn + mp$.
- We used multiplication to define fractions p/q as pairs (p, q) of natural numbers, building the cancelation law $(np)/(nq) = p/q$ into the definition. We extended addition and multiplication to fractions.
- We defined the integers (by taking a 'positive' and a 'negative' copy of each natural number, and throwing in zero). We used inequalities to define integer addition.

To complete the story of arithmetic, there are just a few things remaining. You can check that subtraction of integers is always possible: for integers a and b, $a - b$ is the unique integer such that $(a - b) + b = a$. You can extend multiplication to integers. You can prove the associative, commutative and distributive laws for integers (rather than natural numbers). We can distinguish integers as positive, negative or zero, and use this to define inequality of integers (namely: $a > b$ means that $a - b > 0$).

Finally, we can introduce the system of rational numbers (that is, fractions which may be positive, negative or zero).

Definition 1 A rational number p/q is a pair (p,q) of integers, where $q \neq 0$. For any integer $n \neq 0$, we consider the pair (np,nq) to define the same rational number $(np)/(nq)$ as p/q.

Notice that this is just like the definition of a fraction, invoking pairs of integers instead of pairs of natural numbers. One can add and multiply rational numbers (all the usual properties hold); and one has a notion of inequality for rational numbers. We'll omit the details of these, since they closely resemble things we've already looked at. (Can you fill in the details of the missing definitions?)

We'll now move on from arithmetic, and begin a serious study of inequalities. This will be vital when we come to study limits and calculus.

Absolute values

The absolute value of a rational number x is denoted by $|x|$, where this symbol is defined as follows:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Notice that $|-x| = |x|$ and that $|x-y| > 0$ unless $x = y$. One can think of $|x-y|$ as the 'distance' from x to y.

Theorem 2 *For any rational numbers x and y, we have*

$$|x+y| \leq |x| + |y|.$$

The following lemma (or 'helping theorem') will be useful.

Lemma 3 *If a and b are rational $a \geq 0$, and $b \geq 0$, then*

$$a \leq b \Leftrightarrow a^2 \leq b^2.$$

(Note: when we write $A \Leftrightarrow B$ we mean that when A is true, so is B; and when B is true, so is A.)

Proof Let a and b be non-negative rational numbers. In general, when $x \leq y$ and $z \geq 0$, we have $zx \leq zy$. If $a \leq b$ we can apply this twice to get

$$a^2 \leq ab = ba \leq b^2.$$

Now suppose $a^2 \leq b^2$. We want to prove that $a \leq b$. If this were not the case, then we would have $a > b$. But then $a^2 > b^2$ (prove this!). This contradicts our assumption that $a^2 \leq b^2$. So it must be true that $a \leq b$. □

Proof of the theorem We'll prove the squared version of this inequality:

$$(\star) \qquad (x+y)^2 \le (|x|+|y|)^2.$$

This will do the trick, because we can apply the lemma to the non-negative numbers $a = |x+y|$ and $b = |x|+|y|$ to see that if $(\star)$ is true then $|x+y| \le |x|+|y|$.

Well, $(x+y)^2 = x^2+y^2+2xy$, while $(|x|+|y|)^2 = x^2+y^2+2|x||y|$. We have

$$\begin{aligned} x^2+y^2+2xy &\le x^2+y^2+2|x||y| && \text{since } xy \le |x||y| \\ &= |x|^2+|y|^2+2|x||y| \\ &= (|x|+|y|)^2. \end{aligned}$$

This proves $(\star)$. □

Corollary 4 *For all rational numbers u and v, we have*

$$|u-v| \le |u|+|v|.$$

Proof Let's try to deduce this from the inequality we've already proved. Taking $x = u$ and $y = -v$, that inequality says that

$$|u+(-v)| \le |u|+|-v|.$$

But since $u+(-v) = u-v$ and $|-v| = |v|$, this is exactly what we want to prove. □

Corollary 5 *For all rational numbers x, y and z we have*

$$|x-z| \le |x-y|+|y-z|.$$

The proof is another application of our basic inequality, and is left to you.