Research Methods in Mathematics
Lecture 7: Rational numbers; absolute values

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In this lecture, we'll be working with rational numbers. The system of all rational numbers is denoted by $\mathbb{Q}$ (for quotient—an old-fashioned name for a fraction).

Notation. We'll use the set-theoretic notation $x \in \mathbb{Q}$ to mean that x is a rational number. We also write $A \subset \mathbb{Q}$ to mean that A is a set (i.e., a collection) of natural numbers.

If $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$ then exactly one of the following three things holds: $x = y$, $x < y$, or $x > y$. We haven't proved this, but in fact it follows from the corresponding theorem for natural numbers.

Upper bounds

Definition 1 If $A \subset \mathbb{Q}$, an *upper bound* for A is a number x such that $x \geq a$ for all $a \in A$. We say A is *bounded above* if an upper bound exists. Similarly, a *lower* bound for A is a number y such that $y \leq a$ for all $a \in A$. We say A is *bounded below* if a lower bound exists.

Example 2 Let $A = [0, 1]$. That is, A is the set of all natural numbers x such that $0 \leq x \leq 1$. An upper bound for A is $x = 2$; another is $x = 1$. A lower bound is $x = -87$; another is $x = 0$.

Example 3 If $A = \{1, 2, 3, \dots\}$ then A is bounded below (by 1) but A is not bounded above.

Example 4 If $A = \mathbb{Q}$ then A is not bounded below, nor above.

Example 5 If $A = \{1, 1/2, 1/3, 1/4, \dots\}$ then 1 is an upper bound for A, while 0 is a lower bound.

The following definition is very important.

Definition 6 A *least upper bound* or *supremum* for A is a number $x \in \mathbb{Q}$ in R such that (i) x is an upper bound for A; and (ii) if x' is another upper bound for A then $x' \geq x$. If a supremum exists, it is denoted by $\sup A$.

Example 7 If $A = [0, 1]$ then 1 is a least upper bound for A. Indeed, 1 is an upper bound for A, and if $x < 1$ then x cannot be an upper bound for A (because then either $x < 0$ (so x is not an upper bound because $0 \in A$), or $0 \leq x < 1$ in which case $x \in A$ and $1 > x$, so x is not an upper bound).

Example 8 If $A = \{1, 2, 3, \dots\}$ then A has no upper bound, hence no least upper bound. The same is true of $A = \mathbb{Q}$.

Example 9 Let $A = \{-1, -1/2, -1/3, -1/4, \dots\}$. Then a least upper bound for A is 0. (Note that 0 is not in A!) Indeed, all the numbers in A are negative, so 0 is an upper bound. If $x < 0$ then I claim that x cannot be an upper bound for A. Say $x = -p/q$ with p and q natural numbers. Then

$$x = -\frac{p}{q} \leq -\frac{1}{q} < -\frac{1}{2q}.$$

But $-1/(2q)$ is in A. So x is not an upper bound!

Lemma 10 *If x and x' are both least upper bounds for A then $x = x'$.*

Proof If $x > x'$ then x' is not a least upper bound. So $x \leq x'$. For the same reason, $x' \leq x$. Hence $x = x'$. □

Because of this lemma, we can talk about 'the' least upper bound (or supremum)—there can't be two of them. We write $\sup A$ for the supremum of A.

Similarly, we define a *greatest lower bound* or *infimum* for A to be a number $y \in \mathbb{Q}$ such that (ii) y is a lower bound for A; and (ii) if y' is another lower bound for A then $y' \leq y$. If an infimum exists, it is denoted by $\inf A$. Just as with least upper bounds, there is at most one greatest lower bound.

It may be that no supremum exists, even when an upper bound exists:

Proposition 11 *Let $A = \{a \in \mathbb{Q} : a^2 < 2\}$. Then A has an upper bound but no least upper bound.*

Proof An upper bound for A is given by $x = 10$. To see this, observe that if $a^2 < 2$ then certainly $a^2 \leq 100$. But this implies (see last lecture) that either $a < 0$ or that $a < 10$, so 10 is an upper bound.

We now show that there's no last upper bound. Suppose x were a least upper bound. We'll show that this assumption is untenable.

We can legitimately assume that $x > 0$, since 0 is not an upper bound for A.

It cannot be the case that $x^2 < 2$, for if it were then we could add a small positive number ϵ to x and have $(x+\epsilon)^2 < 2$, contradicting the upper bound property. This is intuitively obvious, but really we ought to prove it. I'll assume it for now, and prove it afterwards.

It cannot be the case that $x^2 > 2$, for if it were then we could subtract a small positive number and still have $(x-\epsilon)^2 > 2$. (Again, I'll defer the details till afterwards.) But then $x - \epsilon$ would be an upper bound (why?) smaller than x.

So the only possibility is that $x^2 = 2$. But there is no rational number x with $x^2 = 2$: this is a famous theorem of Euclid, whose proof of it is the subject of an exercise in Homework 4. □

To be really precise about this, we should show that given any $x > 0$ such that $x^2 < 2$, we can find an $\epsilon > 0$ such that $(x+\epsilon)^2 < 2$. To help us with this, let's note that whatever ϵ we choose, we have

$$(x+\epsilon)^2 = x^2 + 2\epsilon x + \epsilon^2.$$

One useful trick will be to choose ϵ such that $\epsilon \leq x$. Then we have

$$(x+\epsilon)^2 = x^2 + 2\epsilon x + \epsilon^2 \leq x^2 + 2\epsilon x + \epsilon x = x^2 + 3\epsilon x < x^2 + 4\epsilon x.$$

I made the last step to 'make space'—to make sure that I had a strict inequality $<$. Now if I put $\epsilon = \frac{2-x^2}{4x}$ then I'll have

$$x^2 + 4\epsilon x = 2.$$

The only danger is that perhaps this ϵ isn't $\leq x$. So let me choose ϵ to be either x or $\frac{2-x^2}{4x}$, whichever is smaller. Then

$$(x+\epsilon)^2 < x^2 + 4\epsilon x \leq x^2 + 4x\frac{2-x^2}{4x} + 2 = 2,$$

so $(x+\epsilon)^2 < 2$, as desired.

It's left to you to work out how to find an ϵ such that $(x-\epsilon)^2 > 2$ under the assumption that $x^2 > 2$.