Research Methods in Mathematics
Lecture 8: Real numbers

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Completeness

The system rational numbers $\mathbb{Q}$ has many useful properties:

- We can add rational numbers. Addition is associative and commutative. There's a rational number 0 which is 'neutral' for additon: $0+x=x=x+0$. For any x, there's a rational number $-x$ such that $(-x)+x=0$.
- We can multiply rational numbers. Multiplication is associative and commutative. There's a rational number 1 which is neutral for multiplication: $1.x = x = x.1$. For any $x \neq 0$, there's a rational number x^{-1} such that $x^{-1}x = 1$.
- Multiplication distributes over addition.
- There's a notion of inequality for rational numbers, such that any two rational numbers a and b satisfy exactly one of the following: $a<b$, $a=b$, $b<a$. If $a<b$ and $b<c$ then $a<c$.
- Inequality is 'compatible' with addition and multiplication: If $a<b$ and $a'<b'$ then $a+a'<b+b'$. If $a<b$ and $c>0$ then $ac<bc$.

A system of 'numbers' which has notions of addition, multiplication and inequality satisfying all these properties is called an *ordered field*. So far, $\mathbb{Q}$ is our only example of an ordered field.

The rationals have one quite subtle deficiency. As we saw last time, there are some sets of rationals which have an upper bound but do not have a least upper bound. The set A of rationals x with $x^2<2$ has this property: a least upper bound would have to be a square root of 2, but there is no rational square root of 2. Intuitively, it seems that the rational numbers have a 'hole' where $\sqrt{2}$ should be.

A *complete* ordered field is, informally, an ordered field with no holes.

Definition 1 An ordered field R is *complete* if any non-empty subset A of R which has an upper bound has a least upper bound.

It turns out that it possible to 'complete' the rationals.

Theorem 2 *There exists a complete ordered field $\mathbb{R}$, called the system of real numbers. Any other complete ordered field is in 1-1 correspondence with $\mathbb{R}$ in a way that respects addition, multiplication and inequality. The rationals $\mathbb{Q}$ are contained in $\mathbb{R}$, in a way that respects addition, multiplication and inequality.*

We shall not prove this theorem. In fact, we will never answer the question of what a real number 'is' (though this question can be answered—you can read about this in Spivak). We will instead focus on how real numbers *work*. The way they work is that they satisfy arithmetic rules (and rules for inequality) just like the rational numbers, but they have the additional property that every non-empty, bounded-above set S of reals has a least upper bound $\sup S$.

For example, the set $A = \{x \in \mathbb{Q} : x^2 < 2\}$ does not have a rational least upper bound. But $s := \sup A$ exists as a real number. This real number s satisfies $s^2 = 2$. So there is a real square root of 2.

The most familiar representation of real numbers is as decimals. We take some possibly infinite decimal like

$$738.8926492302023....$$

Abstractly, this is just a sequence

$$a_1 a_2 \dots a_m . a_{m+1} a_{m+2} a_{m+3} \dots$$

where each a_j is a digit from 0 to 9 (or the same thing with a $-$ sign at the front). This sequence determines a set S of rational numbers, which in the example would be

$$\{738,\ 738.8,\ 738.89,\ 738.892, \dots\},$$

and in general would be

$$\{a_1 a_2 \dots a_m, a_1 a_2 \dots a_m . a_{m+1}, a_1 a_2 \dots a_m . a_{m+1} a_{m+2}, \dots\}.$$

The set S is bounded above: in the example, it's bounded above by 739, and in general by taking the integer part $a_1 a_2 \dots a_m$ and adding 1. If we stick with rational numbers, the set S may or may not have a least upper bound. If we work with real numbers, there is definitely a least upper bound $\sup S$. So an infinite decimal determines a real number.

Example 3 Consider the infinite decimal $0.999999\dots$. The approximations 0, 0.9, 0.99, etc., form a set of rationals S. We have $\sup S = 1$. So, as a real number, $0.9999999\cdots = 1$.

One way to construct the system of real numbers is to declare that a real number is an infinite decimal

$$\pm a_1 a_2 \ldots a_m . a_{m+1} a_{m+2} a_{m+3} \ldots ,$$

where, to avoid redundancy, we disallow sequences that ends in $99999\ldots$. You can then define addition and multiplication by using the rules of decimal arithmetic.

But it turns out that there are much neater ways of achieving the same result.