Research Methods in Mathematics
Lecture 9: Functions and limits

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Functions and limits

Last time, we introduced the system $\mathbb{R}$ of real numbers. A sightly informal definition of 'function' goes as follows. A *function* f is a rule which assigns to certain real numbers x a real number $f(x)$. The number $f(x)$ depends on x, but for a given x there is only value $f(x)$.

The *domain* of f is the set of real numbers x for which $f(x)$ is defined. So a function gives a single output for each input within the domain.

Example 1 The rule

$$f(x) = \begin{cases} x^2 - 3 & \text{if } x > 7 \\ 4 - x^4 & \text{if } x \leq 7. \end{cases}$$

defines a function whose domain is $\mathbb{R}$.

Example 2 The rule

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

defines another function whose domain is $\mathbb{R}$. The rule

$$s(x) = \sqrt{x} \quad \text{if } x \geq 0$$

defines a function whose domain is the non-negative reals $[0, \infty)$. But

$$t(x) = a \quad \text{if } a^2 = x$$

does not define a function, because, for instance, 1 and -1 are both candidates for $t(1)$.

We could dwell longer on functions and their formal definition. But we have bigger fish to fry: we want to understand the meaning of *limits*.

Limits

Definition 3 Let f be a function defined for all numbers x close to a, except possibly a itself. We say $f(x)$ approaches L as x approaches a, and write

$$f(x) \to L \text{ as } x \to a,$$

if the following is true. Given any number $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.$$

When $f(x) \to L$ as $x \to a$, we call L the limit of f as x approaches a. (Calling it 'the' limit suggests that there can only be one. This is indeed so, as we'll see later.)

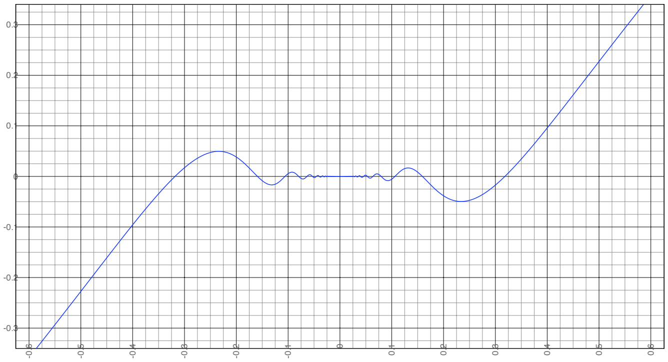
We also write [1]

$$\lim_{x \to a} f(x) = L.$$

Remark One clarification of the definition is in order: the number δ is supposed to be small enough that every point x satisfying $|x - a| < \delta$, except perhaps a itself, lies in the domain of f.

What does this complicated definition mean?

You can think of it as a game. Say $f(x) = x^2 \sin(x^{-1})$ for $x \neq 0$. (This formula doesn't make sense at $x = 0$.) The graph of f is plotted below.



I assert that $f(x) \to 0$ as $x \to 0$.

You say: OK, so find a δ so that when x is within δ of 0, $f(x)$ is within 0.04 of 0. To back up my claim, I then have to show that there's some δ such that for any x with

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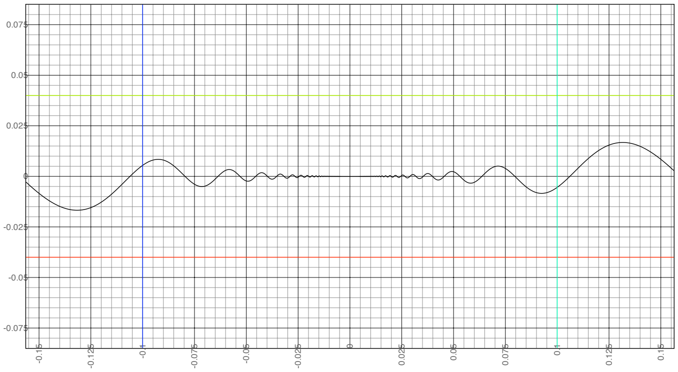
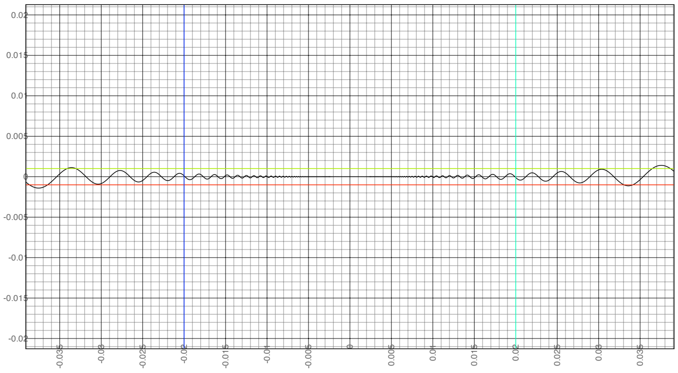
[1] The profusion of names and notations attests to the importance of this notion.

$0 < |x - 0| < \delta$, we have $|f(x)| < 0.04$, i.e. $-0.04 < f(x) < 0.04$. It turns out that $\delta = 0.1$ does the trick: if $-0.1 < x < 0.1$ then $|f(x)| < 0.04$, as you can see in the next figure.



You now say: find a δ so that whenever x is within δ of 0, $f(x)$ is within 0.001 of 1. My original choice of $\delta = 0.1$ no longer works. I need to specify a smaller one. It turns out that $\delta = 0.02$ now works: if $-0.02 < x < 0.02$ then $|f(x)| < 0.001$.



And so on.

Graphical interpretation of limits. If you mark two horizontal lines at $y = L - \epsilon$ and $y = L + \epsilon$, I can find two vertical lines at $x = a - \delta$ and $y = a + \delta$ so that, when x lies between these two vertical lines, the graph of f lies between the two horizontal lines.

The terminology "*the* limit" needs a little justification. Can there be two limits as $x \to a$? What if $f(x) \to M$ also?

Lemma 4 *Suppose that* $f(x) \to L$ *and* $f(x) \to M$ *as* $x \to a$. *Then* $L = M$.

Proof For any $\epsilon > 0$, there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta_1$, while $|f(x) - M| < \epsilon$ whenever $0 < |x - a| < \delta_2$. Suppose we take $\delta = \min(\delta_1, \delta_2)$ (i.e. δ is either δ_1 or δ_2—whichever is smaller). Then, if $0 < |x - a| < \delta$, we'll have both $|f(x) - L| < \epsilon$ and $|f(x) - M| < \epsilon$. Note that this is true whatever ϵ we start with, but that the value of δ will depend on ϵ.

Suppose now that $L \neq M$; say $|L - M| = d > 0$. Let's take $\epsilon = d/2$, and find $\delta > 0$ as above. Then, when $0 < |x - a| < \delta$, we will have $|f(x) - L| < d/2$ and $|f(x) - M| < d/2$. This makes no sense! In fact,

$$d = |L - M| = |(f(x) - M) - (f(x) - L)| \leq |f(x) - M| + |f(x) - L| < d/2 + d/2 = d,$$

so $d < d$, which is absurd. So $L = M$. □

Spivak reference: Chapters 3, 5 (4 is also worth a look).