Freshman Research Initiative:
Research Methods in Mathematics

TIMOTHY PERUTZ

UT Austin, Fall Semester, 2010.

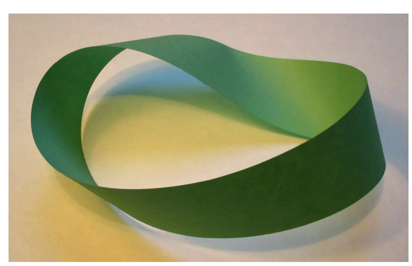
- *Course number:* M 310T.
- *Class meets:* Tuesday, Thursday 9:30–11:00 a.m.
- *Instructor:* Timothy Perutz.
- *Web:* `www.math.utexas.edu/~perutz`

Questions? Email me (`perutz@math.utexas.edu`) or ask in person (RLM 10.136).

Overview

What do research mathematicians *do*? Solve really hard calculus problems involving enormously complicated formulae? Not really. Ask computers to solve equations for them? Only from time to time. What then?

In this course, you'll get a taste of mathematics *as it's understood by mathematicians.*



It's a broader, more creative field than you may realize. It's about explaining patterns and explaining connections between ideas. The course will be divided into three chapters, each giving a flavor of a different aspect of the subject. If you're considering going deeper into this subject, this will give you a flying start.

The mathematics covered in this course may be the most challenging you have seen so far. But it might also be the most satisfying.

In the next three pages, I'll describe the three topics I plan to cover.

THE

ANALYST;

OR, A

DISCOURSE

Addressed to an

Infidel MATHEMATICIAN.

WHEREIN

It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

By the AUTHOR of *The Minute Philosopher.*

First cast out the beam out of thine own Eye; and then shalt thou see clearly to cast out the mote out of thy brother's eye. S. Matt. c. vii. v. 5.

LONDON:

Printed for J. TONSON in the *Strand.* 1734.

Figure 1: The frontispiece to *The Analyst*, a polemic by the philosopher-bishop George Berkeley against what he saw as woolly thinking in the works of Newton and his followers on calculus. Newton's infinitesimal increments, he said, must be "the ghosts of departed quantities".

Today, mathematicians recognize that Berkeley had a good point—but now we know how to dispel the fog!

From counting to calculus. *Mathematics asks for crystal-clear arguments.*

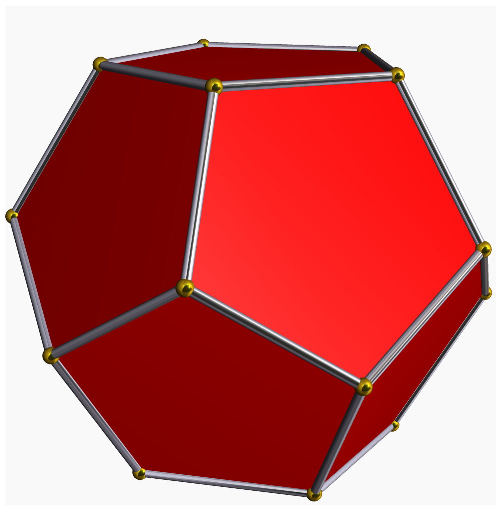
Nothing can be fuzzy, even the idea of a number. In the first part of the course, we'll examine how to go, in logical progression, from the idea of counting to natural numbers, negative numbers, rational and real numbers. Once we understand how real numbers work, we can make differential calculus absolutely precise.

What's jazz? somebody asked Louis Armstrong. 'Man, if you gotta ask you'll never know', he said. What's a limit *really*? people asked the calculus pioneers—Newton, Leibnitz, Euler and Lagrange. If you gotta ask, they said, you'll never know. In the first part of this course you'll find out how mathematicians today answer that question.

These ideas lie beneath a great deal of current mathematical research. By learning about them now, you'll get a headstart in your mathematical studies.

Algebra and geometry of linear maps. *Geometry illuminates algebra, and algebra helps us do geometry.*

Suppose we are writing a 3D computer graphics program. The computer shows an image of a 3-dimensional object, as viewed from a camera pointing in a particular direction.



But now we'd like to be able to understand how the image changes if we rotate the camera.

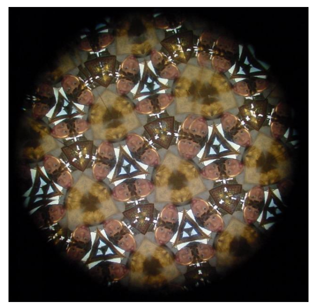
How can we achieve this? We'll see how to solve problems like this using the linear algebra of 3×3 matrices, like this:

$$R = A \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1}.$$

We can breathe life into arrays of numbers by understanding them geometrically. And we can solve geometric problems by encoding them as matrices.

The symmetries of plane patterns. *Modern topology helps us understand symmetry.*

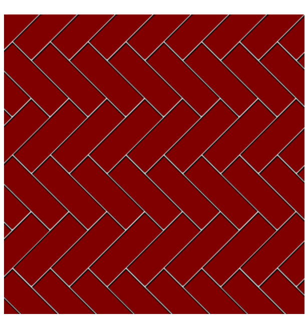
In this part of the course, we'll see that mathematics can be strongly visual, creative and fun. We'll draw on computer graphics and make models with paper, scissors and tape. The plan is to examine tiling patterns and their symmetries. It turns out that there are exactly 17 possible kinds of tiling when we classify them according to their symmetries—neither more nor less!

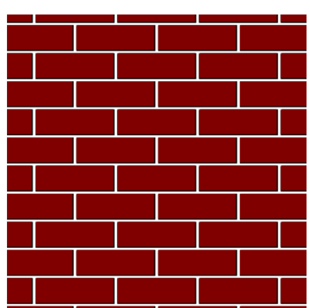


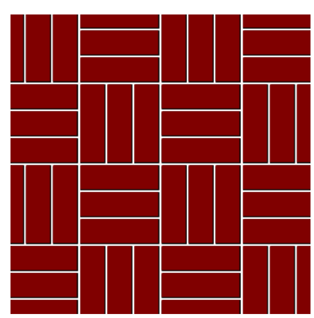
This fact has been known for a while, but we'll study a modern proof inspired by the part of mathematics called *topology*. The idea is to take your pattern and build from it a shape called its *orbifold* which demonstrates the simplest repeating unit. (The Möbius band on the front page is one example of an orbifold.) We can write down the features of the orbifold in a special notation. The pattern of bricks called running bond (the middle one below) is encapsulated in the code $2\star 22$, while the inset kaleidoscopic pattern is written as $\star 632$.

The famous 18th Century mathematician Euler discovered a formula that says that if you look at any polyhedron (cube, tetrahedron, dodecahedron, etc.), the number of corners minus the number of edges plus the number of faces equals 2. (For the dodecahedron on the previous page, $20 - 30 + 12 = 2$.)

It wasn't until much more recently that a similar formula for orbifolds was discovered. It tells us that only seventeen orbifolds can arise from repeating patterns in the plane!







Assessment

There won't be tests or a final exam in this course. Instead, there will be:

- *Homework*. Most weeks, including questions of various sorts. Some will be routine problems to help learn the material. Others will give you practice in writing proofs. A few will be tough nuts which I hope you will enjoy solving.
- *Extended assignments*. There will be three of these—one for each part of the course. You will be able to choose from a short list of titles.

FAQ

- What are the prerequisites?

Fluency in algebra, trigonometry and differential calculus. Enthusiasm. Diagnostic questions: Differentiate the function $f(x) = 1/(x-4)^2$. What is $\sin(2\pi/3)$? What is the simplest way to write $\ln e^2$? Write out the expansion of $(1-x)^3$.

- Is this a research course? If not, why not?

The homeworks will give just a little of the flavor of mathematical research, because some of the questions will require persistence and ingenuity. Mathematical knowledge is cumulative, and to do meaningful research you need an extensive grounding in the subject. If that's something you are considering pursuing, this course will help prepare you.

- Is this course for me?

The answer could be 'yes' if the some of the following apply to you: I like logical arguments. I enjoy solving problems. I like thinking about things for myself. I would like to learn how to prove things for myself. I think about things analytically. I think about things visually. I would like to get a 'bigger picture' of mathematics.

It could be 'no' if the following tend to apply to you: I seldom enjoyed school math. I have not studied calculus. I only care about the answer, not how you get to it. I only care about things where I can immediately see the practical value. I don't like abstraction. I think of proofs as pointless pedantry. I want to minimize the time I spend on math homework.