Freshman Research Initiative:
Research Methods in Mathematics

TIMOTHY PERUTZ

UT Austin, Fall Semester, 2010.

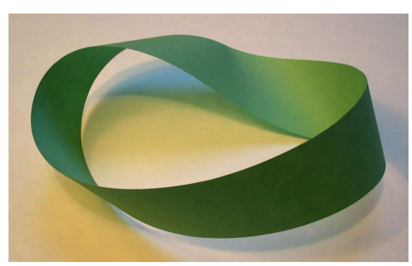
- *Course number:* M 310T.
- *Class meets:* Tuesday, Thursday 9:30–11:00 a.m.
- *Instructor:* Timothy Perutz.
- *Office hours:* Tuesday, Wednesday 4–5, RLM 10.136.
- *T.A.:* Michael Kelly (t.b.c.) `mkelly@math.utexas.edu`, RLM 9.128.
- *Web:* `www.math.utexas.edu/users/perutz/ResMethods2010.html`

Questions? Email me (`perutz@math.utexas.edu`) or ask in person (RLM 10.136).

Course overview

What do research mathematicians *do*? Solve really hard calculus problems involving enormously complicated formulae? Not really. Ask computers to solve equations for them? Only from time to time. What then?

In this course, you'll get a taste of mathematics *as it's understood by mathematicians.*



It's a broader, more creative field than you may realize. It's about explaining patterns and explaining connections between ideas. The course will be divided into three chapters, each giving a flavor of a different aspect of the subject. If you're considering going deeper into this subject, this will give you a flying start.

The mathematics covered in this course may be the most challenging you have seen so far. But it might also be the most satisfying.

In the next three pages, I'll describe the three topics I plan to cover.

THE

ANALYST;

OR, A

DISCOURSE

Addressed to an

Infidel MATHEMATICIAN.

WHEREIN

It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

By the AUTHOR of *The Minute Philosopher.*

First cast out the beam out of thine own Eye; and then shalt thou see clearly to cast out the mote out of thy brother's eye. S. Matt. c. vii. v. 5.

LONDON:

Printed for J. TONSON in the *Strand.* 1734.

Figure 1: The frontispiece to *The Analyst*, a polemic by the philosopher-bishop George Berkeley against what he saw as woolly thinking in the works of Newton and his followers on calculus. Newton's infinitesimal increments, he said, must be "the ghosts of departed quantities".

Today, mathematicians recognize that Berkeley had a good point—but now we know how to dispel the fog!

From counting to calculus. *Mathematics asks for crystal-clear arguments.*

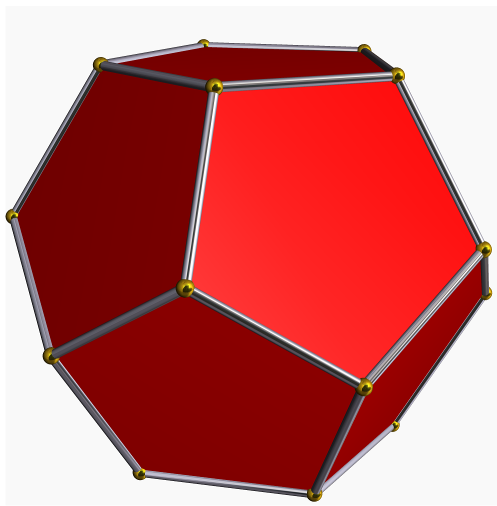
Nothing can be fuzzy, even the idea of a number. In the first part of the course, we'll examine how to go, in logical progression, from the idea of counting to natural numbers, negative numbers, rational and real numbers. Once we understand how real numbers work, we can make differential calculus absolutely precise.

What's jazz? somebody asked Louis Armstrong. 'Man, if you gotta ask you'll never know', he said. What's a limit *really*? people asked the calculus pioneers—Newton, Leibnitz, Euler and Lagrange. If you gotta ask, they said, you'll never know. In the first part of this course you'll find out how mathematicians today answer that question.

These ideas lie beneath a great deal of current mathematical research. By learning about them now, you'll get a headstart in your mathematical studies.

Algebra and geometry of linear maps. *Geometry illuminates algebra, and algebra helps us do geometry.*

Suppose we are writing a 3D computer graphics program. The computer shows an image of a 3-dimensional object, as viewed from a camera pointing in a particular direction.



But now we'd like to be able to understand how the image changes if we rotate the camera.

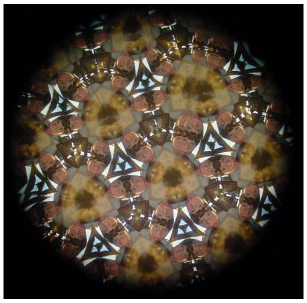
How can we achieve this? We'll see how to solve problems like this using the linear algebra of 3×3 matrices, like this:

$$R = A \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1}.$$

We can breathe life into arrays of numbers by understanding them geometrically. And we can solve geometric problems by encoding them as matrices.

The symmetries of plane patterns. *Modern topology helps us understand symmetry.*

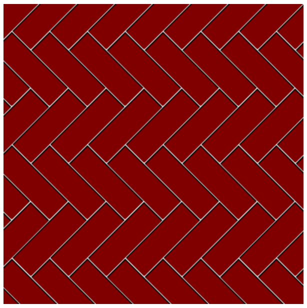
In this part of the course, we'll see that mathematics can be strongly visual, creative and fun. We'll draw on computer graphics and make models with paper, scissors and tape. The plan is to examine tiling patterns and their symmetries. It turns out that there are exactly 17 possible kinds of tiling when we classify them according to their symmetries—neither more nor less!

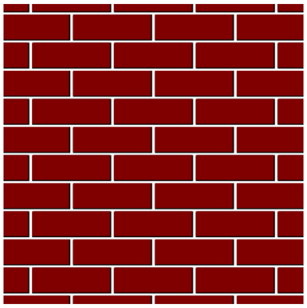


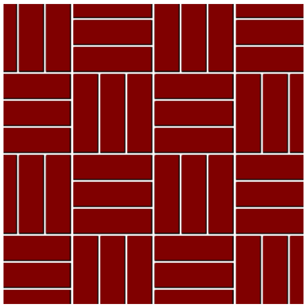
This fact has been known for a while, but we'll study a modern proof inspired by the part of mathematics called *topology.* The idea is to take your pattern and build from it a shape called its *orbifold* which demonstrates the simplest repeating unit. (The Möbius band on the front page is one example of an orbifold.) We can write down the features of the orbifold in a special notation. The pattern of bricks called running bond (the middle one below) is encapsulated in the code $2{\star}22$, while the inset kaleidoscopic pattern is written as ${\star}632$.

The famous 18th Century mathematician Euler discovered a formula that says that if you look at any polyhedron (cube, tetrahedron, dodecahedron, etc.), the number of corners minus the number of edges plus the number of faces equals 2. (For the dodecahedron on the previous page, $20 - 30 + 12 = 2$.)

It wasn't until much more recently that a similar formula for orbifolds was discovered. It tells us that only seventeen orbifolds can arise from repeating patterns in the plane!







Assessment

There won't be tests or a final exam in this course. Instead, there will be:

- *Homework*. Most weeks, including questions of various sorts. Some will be routine problems to help learn the material. Others will give you practice in writing proofs. A few will be tough nuts which I hope you will enjoy solving. There will be ten homeworks, worth a total of 45%. You drop your lowest grade.
- *Extended assignments*. There will be three of these—one for each part of the course. You will be able to choose from a short list of titles. Each will be weighted at 15%.
- *Participation*. 10% of your grade will be for participation. By this I mean attending class and asking or answering questions in class or at my office hours.

FAQ

- What are the prerequisites?

Fluency in algebra, trigonometry and differential calculus. Enthusiasm. Diagnostic questions: Differentiate the function $f(x) = 1/(x-4)^2$. What is $\sin(2\pi/3)$? What is the simplest way to write $\ln e^2$? Write out the expansion of $(1-x)^3$.

- Is this a research course? If not, why not?

The homeworks will give just a little of the flavor of mathematical research, because some of the questions will require persistence and ingenuity. Mathematical knowledge is cumulative, and to do meaningful research you need an extensive grounding in the subject. If that's something you are considering pursuing, this course will help prepare you.

- Is this course for me?

The answer could be 'yes' if the some of the following apply to you:

I like logical arguments. I enjoy solving problems. I like thinking about things for myself. I would like to learn how to prove things for myself. I think about things analytically. I think about things visually. I would like to get a 'bigger picture' of mathematics.

It could be 'no' if the following tend to apply to you: I seldom enjoyed school math. I have not studied calculus. I only care about the answer, not how you get to it. I only care about things where I can immediately see the practical value. I don't like abstraction. I think of proofs as pointless pedantry. I want to minimize the time I spend on math homework.

Textbooks

The course will be self-contained, so textbooks are not absolutely required. However, the following texts are highly recommended for Parts I and II of the course, respectively.

- Michael Spivak, *Calculus*, 4th edition. Cambridge University Press, 2008. Currently available for $85. A brilliantly written text on analysis—a different kettle of fish from other books with the same title. A solution manual is available.
- Alexander Givental, *Linear algebra and differential equations*, Berkeley Mathematics Lecture Notes Vol 11. American Mathematical Society, 2001. Currently available for $21. Tersely written—read it slowly!—this book goes straight to what's important. We will cover a fraction of what's here.

There is no need to buy a text for Part III, but if you can afford it, this would be the one to go for:

- J. Conway, H. Burgiel and C. Goodman-Strauss, *The Symmetries of Things*. A. K. Peters, 2008. Wonderful pictures; presents the orbifold method. Currently available for $60.

For further reading about mathematics generally, I recommend the following.

- T. Gowers, *Mathematics: A Very Short Introduction*. Oxford University Press, 2002. Currently available for less than $10! Brevity and price are virtues of this book, but so is its quality. To get an impression of what mathematicians actually do, this may be the best place to start.
- T. Körner, *The Pleasures of Counting*. Cambridge University Press, 1996. Opinionated and humorously written essays on mathematics as applied to the real world. Currently available for $70.

FAQ

- What are the prerequisites?

Fluency in algebra and trigonometry. Exposure to differential calculus. Enthusiasm. Diagnostic questions: What is the derivative of the function $f(x) = 1/(x-4)^2$? What is $\sin(2\pi/3)$? What is the simplest way to write $\ln e^2$? Write out the expansion of $(1-x)^3$.

- Is this a research course? If not, why not?

You won't get to do any research in this class, but the homeworks will give just a little of the flavor of mathematical research, because some of the questions will require persistence and ingenuity. Mathematical knowledge is cumulative, and to do meaningful research you need an extensive grounding in the subject. If that's something you are considering pursuing, this course will begin to prepare you.

- Is this course for me?

The answer could be 'yes' if the some of the following apply to you: I enjoyed studying calculus in school. I like logical arguments. I enjoy solving problems. I like thinking about things for myself. I would like to learn how to prove things for myself. I think about things analytically. I think about things visually. I would like to get a 'bigger picture' of mathematics.

It could be 'no' if the following tend to apply to you: I seldom enjoyed school math. I have not studied calculus. I only care about the answer, not how you get to it. I only care about things where I can immediately see the practical value. I don't like abstraction. I think of proofs as pointless pedantry. I want to minimize the time I spend on math homework.

- What is the grading policy?

Your numerical scores from homeworks and tests will be combined to give a weighted average. At the end of the semester, I will convert this into a (plus/minus) letter grade. When I do this, I will take account of the difficulty of the course. If you are used to getting 90% or more on every math assignment, you should prepare for some lower numerical scores—but NOT necessarily a lower letter grade.

Class plans

Part I: From counting to calculus

- Introduction: The ghosts of departed quantities? Berkeley's criticism of Lebniz and Newton.
- Counting and induction. The properties of natural numbers and the principle of induction.
- Addition, multiplication and inequality.
- Rational numbers and upper bounds
- The real numbers
- Functions and their limits.
- Continuity.
- The intermediate value theorem.
- Differentiation. Positive derivative implies strictly increasing.
- Differentiation rules.
- "Plus a constant": the uniqueness of antiderivatives, deduced from IVT via the C^1 mean-value theorem.

Part II: Algebra and geometry of linear maps

- Vectors and matrices in 2 dimensions.
- Linear maps and matrices.
- Orthogonal transformations. Rotations and reflections.
- Classifying orthogonal transformations. Rotations and reflections of the plane.
- Vectors and matrices in three dimensions.
- Eigenvalues and eigenvectors.
- Classifying orthogonal transformations in three dimensions.

Part III: Symmetry patterns

- Classifying rigid transformations of the plane.
- Rosette patterns. The seven frieze patterns. Their signatures.
- Wallpaper patterns. The kaleidoscopic and gyroscopic patterns.

- The remaining wallpaper patterns. Their orbifolds. Recognizing them.
- The classification theorem via Conway's magic theorem.
- Euler characteristics.
- Proving the magic theorem.

Due dates

Homework: each Thursday, at the beginning of class, starting September 2, except when an extended assignment is due. Extended assignments: October 5, November 2, December 2.

Policies

This will be quite a small class, and I hope it will be one with plenty of participation from all the students in it. To make this work, I expect you to adhere to some common-sense rules: don't skip class (illness excepted); arrive on time; put your phone and other gadgets away (definitely no texting, instant messaging, etc.); take notes in class.

You must adhere to the usual academic standards of the University of Texas at Austin. You can discuss your homework with others, but to avoid any hint of plagiarism, you should write it up alone.

If you have (or think you might have) a disability relevant to this class, you can request accommodations from the Division of Diversity and Community, Services for Students with Disabilities. If there's something I can do differently that will help, I'll gladly discuss it with you.

If you need to miss class for a religious holiday, university policy asks that you let me know two weeks in advance.