Research Methods in Mathematics, Lecture 1

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Part I: From counting to calculus

The ghosts of departed quantities?

In a 1734 pamphlet entitled *The Analyst*, the philosopher George Berkeley argued that infinitesimal calculus, the subject that had been developed independently by Newton and Leibniz in the 1660s and 1670s that we usually now just call calculus, was founded on sand. Berkeley was also a bishop, and the abstract of his article reads as follows:

> It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

He argues that the answer is 'no'. His chief concern, which is a valid one, is that in the process of differentiation, one invokes a small but non-zero increment, which in Leibniz's notation would be written Δx and in Newton's as o. But then one changes the rules, transmuting Δx into a 'infinitesimal' increment dx, smaller than any ordinary increment—which is apparently to say, no increment at all. In Berkeley's sarcastic words, infinitesimal increments are "the ghosts of departed quantities". We will let him speak for himself:

> Let the Quantity x flow uniformly, and be it proposed to find the Fluxion [*derivative*] of x^n. [...] I suppose that the Quantity x flows, and by flowing is increased, and its Increment I call o, so that by flowing it becomes $x + o$. And as x increaseth, it follows that every Power of x is likewise increased in a due Proportion. Therefore as x becomes $x + o$, x^n will become $\overline{x+o}|^n$ [*i.e.,* $(x+o)^n$]: that is, according to the Method of infinite Series,[1]
>
> $$x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2} + \&c.$$

[1] If n is a positive integer, which will suffice for Berkeley's point, this is a finite binomial series, not an infinite one, and can be justified much more simply.

> And if from the two augmented Quantities we subduct the Root and the Power respectively [*i.e. we consider* $(x+o)-x$ *and* $(x+o)^n - x^n$], we shall have remaining the two Increments, to wit,
>
> $$o \text{ and } nox^{n-1} + \frac{nn-n}{2}oox^{n-2} + \&c.$$
>
> which Increments, being both divided by the common Divisor o, yield the Quotients
>
> $$1 \text{ and } nx^{n-1} + \frac{nn-n}{2}ox^{n-2} + \&c.$$
>
> which are therefore Exponents of the Ratio of the Increments. Hitherto I have supposed that x flows, that x hath a real Increment, that o is something. And I have proceeded all along on that Supposition, without which I should not have been able to have made so much as one single Step. From that Supposition it is that I get at the Increment of x^n, that I am able to compare it with the Increment of x, and that I find the Proportion between the two Increments. I now beg leave to make a new Supposition contrary to the first, *i. e.* I will suppose that there is no Increment of x, or that o is nothing; which second Supposition destroys my first, and is inconsistent with it, and therefore with every thing that supposeth it. I do nevertheless beg leave to retain nx^{n-1}, which is an Expression obtained in virtue of my first Supposition, which necessarily presupposeth such Supposition, and which could not be obtained without it: All which seems a most inconsistent way of arguing, and such as would not be allowed of in Divinity.

Berkeley does not doubt that Leibniz and Newton had found something important. His quarrel is with their logic:

> I have no Controversy about your Conclusions, but only about your Logic and Method.[...] It must be remembred that I am not concerned about the truth of your Theorems, but only about the way of coming at them; whether it be legitimate or illegitimate, clear or obscure, scientific or tentative.

The great mathematicians of the eighteenth and early nineteenth centuries all had their own ways of explaining why calculus works. But most of their arguments reach a point where they become vague and unconvincing. The language of the following quotation from Lagrange (from as late 1811) precisely bears out Berkeley's complaint:

> When we have grasped the spirit of the infinitesimal method, and have verified the exactness of its results either by the geometrical method of

> prime and ultimate ratios, or by the analytical method of derived functions, we may employ infinitely small quantities as a sure and valuable means of shortening and simplifying our proofs.

The fact is that, until the nineteenth century, mathematicians had higher priorities than making calculus rigorous. In the early nineteenth century—prompted by the need to teach calculus to students—Cauchy precisely formulated the notion of a limit, and hence that of a derivative. A that time, mathematics was just beginning to undergo what would be a radical shift in philosophy. Up to that point, it had been the natural science of numbers and geometry. By the end of the nineteenth century, mathematics was being viewed as the logical deduction of the consequences of certain axioms; consequences which may or may not be applicable to real-world phenomena. As part of this shift of perspective, Dedekind, Weierstrass and others showed that one can trace a connected logical path from the notion of 'natural number' to that of 'real number', and from there to notions of 'limit', 'continuous function' and 'derivative'.

In the first part of this course, we shall trace this line of thought, and thereby see how differential calculus can be logically grounded. The techniques involved in this development are usually called "mathematical analysis", or just "analysis". Mastering analysis is a skill that can be developed, just as one can learn to differentiate and integrate.

Analysis does not invoke any magical new principle. It is based on nothing fancier than the observation that some numbers are bigger than others.[2]

[2] In the 1960s, Abraham Robinson developed a completely different way of developing calculus, called 'non-standard analysis', in which infinitesimals play an important part. To set up a number system which has infinitesimals (the 'hyperreal numbers') Robinson digs into the structure of mathematical logic. Though correct, his method looks like smoke and mirrors.