Research Methods in Mathematics
Lecture 8: The intermediate value theorem

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The intermediate value theorem

Theorem 1 (Intermediate value theorem) *Suppose that f is a continuous function on the interval $[a, b]$. Suppose that $f(a) < 0$ and $f(b) > 0$. Then there is some x in (a, b) with $f(x) = 0$.*

This statement shows that our definition of the real numbers is on the right track. If we work not with real numbers but with the rational numbers $\mathbb{Q}$, the intermediate value theorem fails: Define a function f by setting $f(x) = -1$ if $x^2 < 2$, and $f(x) = 1$ if $x^2 > 2$. This is a continuous function on $\mathbb{Q}$ (after all, where could it be discontinuous—only at $\pm\sqrt{2}$, but those are not points in the domain). It gets from -1 to 1 without ever taking the value 0.

Proof Let $A = \{t \in [a, b] : f(t) \leq 0\}$. Then A is non-empty (because $a \in A$). It is bounded above (by b). So, by the completeness axiom, A has a supremum $x = \sup A$. I claim that $f(x) = 0$.

Let $y = f(x)$. To prove the claim, we must rule out two possibilities: (i) that $y < 0$, and (ii) that $y > 0$. Suppose first that $y < 0$. Let $\epsilon = -y/2$. Then there exists $\delta > 0$ such that, when $0 < |x' - x| < \delta$, we have $|f(x') - f(x)| < \epsilon$. Let $x' = x + \delta/2$. Then $0 < |x' - x| < \delta$, so $|f(x') - y| < \epsilon$, and hence $f(x') < y + \epsilon = y/2 < 0$. But this says that $x' \in A$, contradicting the fact that it is bigger than the supposed upper bound x.

Now suppose that $y > 0$. Let $\epsilon = y/2$. Then there exists $\delta > 0$ such that, when $0 < |x' - x| < \delta$, we have $|f(x') - f(x)| < \epsilon$. Then for any $x' \in [x - \delta, x]$, we will have $|f(x') - y| < \epsilon$, and hence $f(x') > y - \epsilon = y/2 > 0$. Hence $x - \delta/2$ is an upper bound for A, contradicting the fact that x is the least upper bound.

This leaves only the possibility that $y = 0$, and hence finishes the proof. □

Corollary 2 *Every positive real number a has a real square root.*

Proof □

Corollary 3 *Let p be a polynomial of odd degree and leading coefficient 1, say*

$$p(x) = x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0.$$

Then there exists a real number x with $p(x) = 0$.

Proof I claim that for x large enough, $p(x) > 0$. Indeed, suppose $x > 1$. Then

$$\begin{aligned} |a_{2n}x^{2n} + \cdots + a_1 x + a_0| &\leq |a_{2n}|x^{2n} + |a_{2n-1}|x^{2n-1} + \cdots + |a_0| \\ &\leq |a_{2n}|x^{2n} + |a_{2n-1}|x^{2n} + \cdots + |a_0|x^{2n} \\ &= |a_{2n} + |a_{2n-1}| + \cdots + |a_0||x^{2n}. \end{aligned}$$

Put $C = 1 + |a_{2n} + |a_{2n-1}| + \cdots + |a_0||$. Then, when $x > C$,

$$p(x) \geq x^{2n+1} - |a_{2n}x^{2n} + \cdots + a_1 x + a_0| \geq x^{2n+1} - Cx^2 n > x^{2n+1} - x^{2n+1} = 0.$$

This proves the claim. Similarly (!?), when $x < -C$, we have $p(x) < 0$.

Sums and products of continuous functions are continuous. Therefore (by an inductive argument) p is continuous.

So, by the IVT, p has a zero somewhere in $[-C, C]$. □

Spivak reference: Chapter 6, Chapters 7–8 (only one of the ‘hard theorems’).