MATHEMATICAL GAMES
Author(s): Martin Gardner
Source: *Scientific American*, Vol. 236, No. 1 (January 1977), pp. 110-121
Published by: Scientific American, a division of Nature America, Inc.
Stable URL: https://www.jstor.org/stable/10.2307/24953856

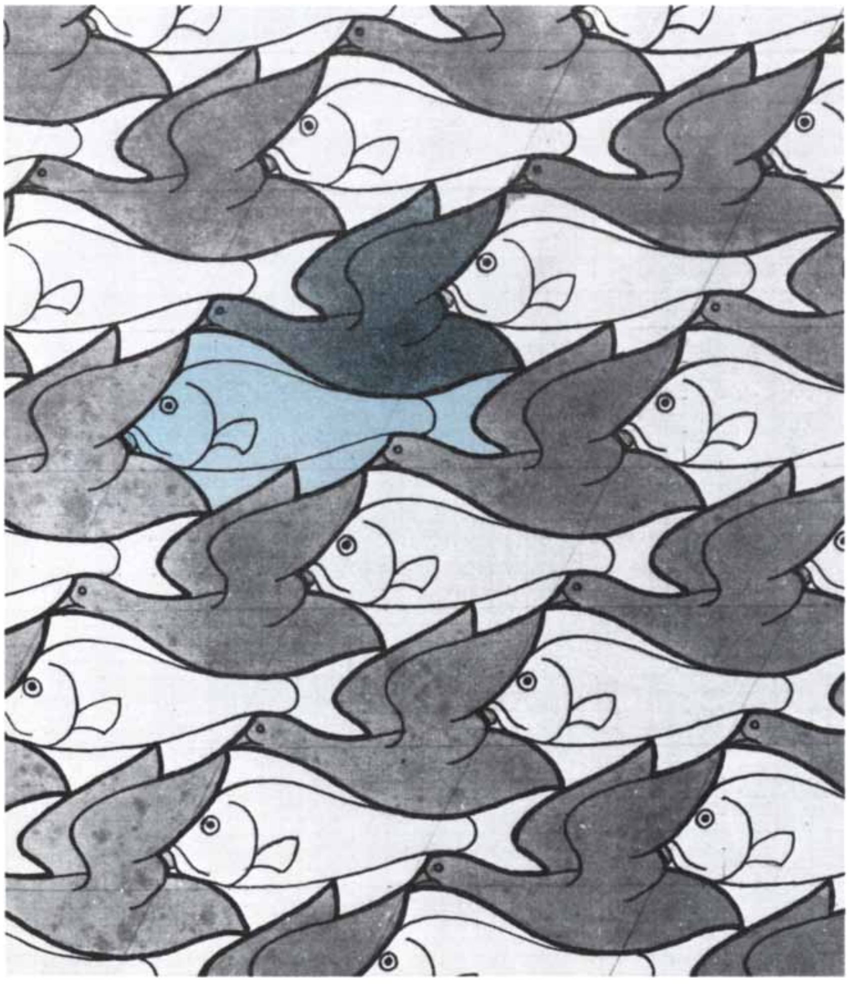
MATHEMATICAL GAMES

Extraordinary nonperiodic tiling that enriches the theory of tiles

by Martin Gardner

In August, 1975, at the end of a two-part article on tiling the plane with congruent convex polygons, I promised a later article on nonperiodic tiling. This column fulfills that promise and presents for the first time a remarkable nonperiodic tiling discovered by Roger Penrose, a British mathematical physicist. First let me give some definitions and background.

A periodic tiling is one on which you can outline a region that tiles the plane by translation, that is, by shifting the position of the region without rotating or reflecting it. M. C. Escher, the Dutch artist, is famous for his many pictures of periodic tilings with shapes that resemble living things. The illustration below is typical. The colored area outlines a fundamental region that tiles by translation. Think of the plane as being covered with transparent paper on which each tile is outlined. Only if the tiling is periodic can you shift the paper, without rotation, to a new position where all outlines again exactly fit.



A periodic tessellation by M. C. Escher (1938)

An infinity of shapes—for instance the regular hexagon—tile only periodically. An infinity of other shapes tile both periodically and nonperiodically. A checkerboard is easily converted to a nonperiodic tiling by identical isosceles right triangles or by quadrilaterals. Simply bisect each square as is shown at the left in the top illustration on the opposite page, altering the orientations to prevent periodicity.

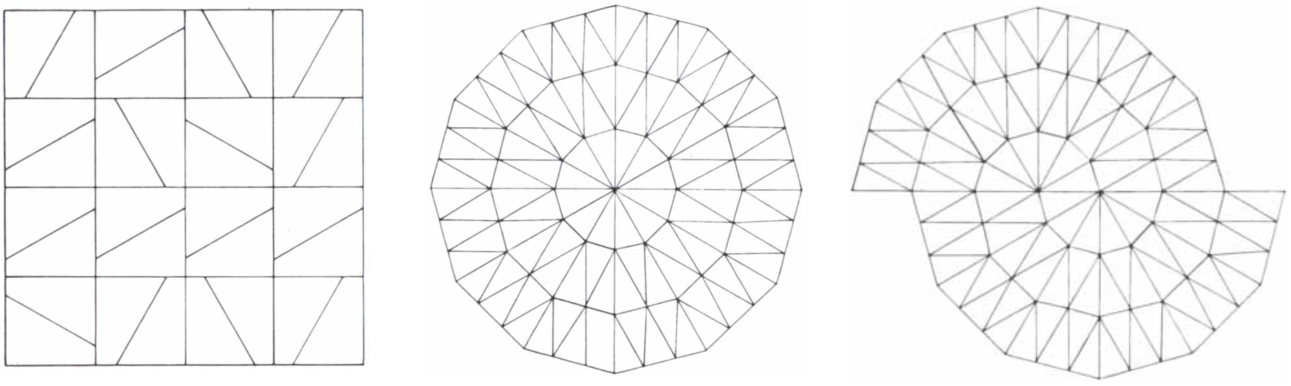
Isosceles triangles also tile in the radial fashion shown in the center of the illustration. Although the tiling is highly ordered, it is obviously not periodic. As Michael Goldberg pointed out in a 1955 paper titled "Central Tessellations," such a tiling can be sliced in half, and then the half planes can be shifted one step or more to make a spiral form of nonperiodic tiling, as is shown at the right in the illustration. The triangle can be distorted in an infinity of ways by replacing its two equal sides with congruent lines as is shown in the middle illustration on the opposite page. If the new sides have straight edges, the result is a polygon of 5, 7, 9, 11... edges that tiles spirally. The bottom illustration on the opposite page shows a striking pattern obtained in this way from a nine-sided polygon. It was first found by Heinz Voderberg in a complicated procedure. Goldberg's method of obtaining it makes it almost trivial.

In all known cases of nonperiodic tiling by congruent figures the figure also tiles perodically. The right part of the middle illustration on the opposite page shows how two of the Voderberg enneagons go together to make an octagon that tiles periodically in an obvious way.

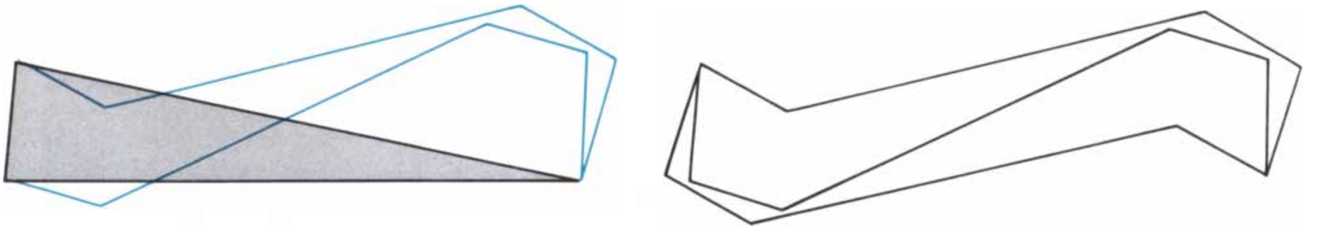
Another kind of nonperiodic tiling is obtained by tiles that group together to form larger replicas of themselves. Solomon W. Golomb calls them "rep-tiles." (See Chapter 19 of my book *Unexpected Hanging*.) The bottom illustration on page 112 shows how a shape called the "sphinx" tiles nonperiodically by giving rise to ever larger sphinxes. Again, two sphinxes (with one sphinx rotated 180 degrees) tile periodically in an obvious way.

Are there sets of tiles, having two or more *different* shapes, that tile only nonperiodically? By "only" we mean that neither a single shape or subset nor the entire set tiles periodically but that by using all of them a nonperiodic tiling is possible. Rotating and reflecting tiles are allowed.

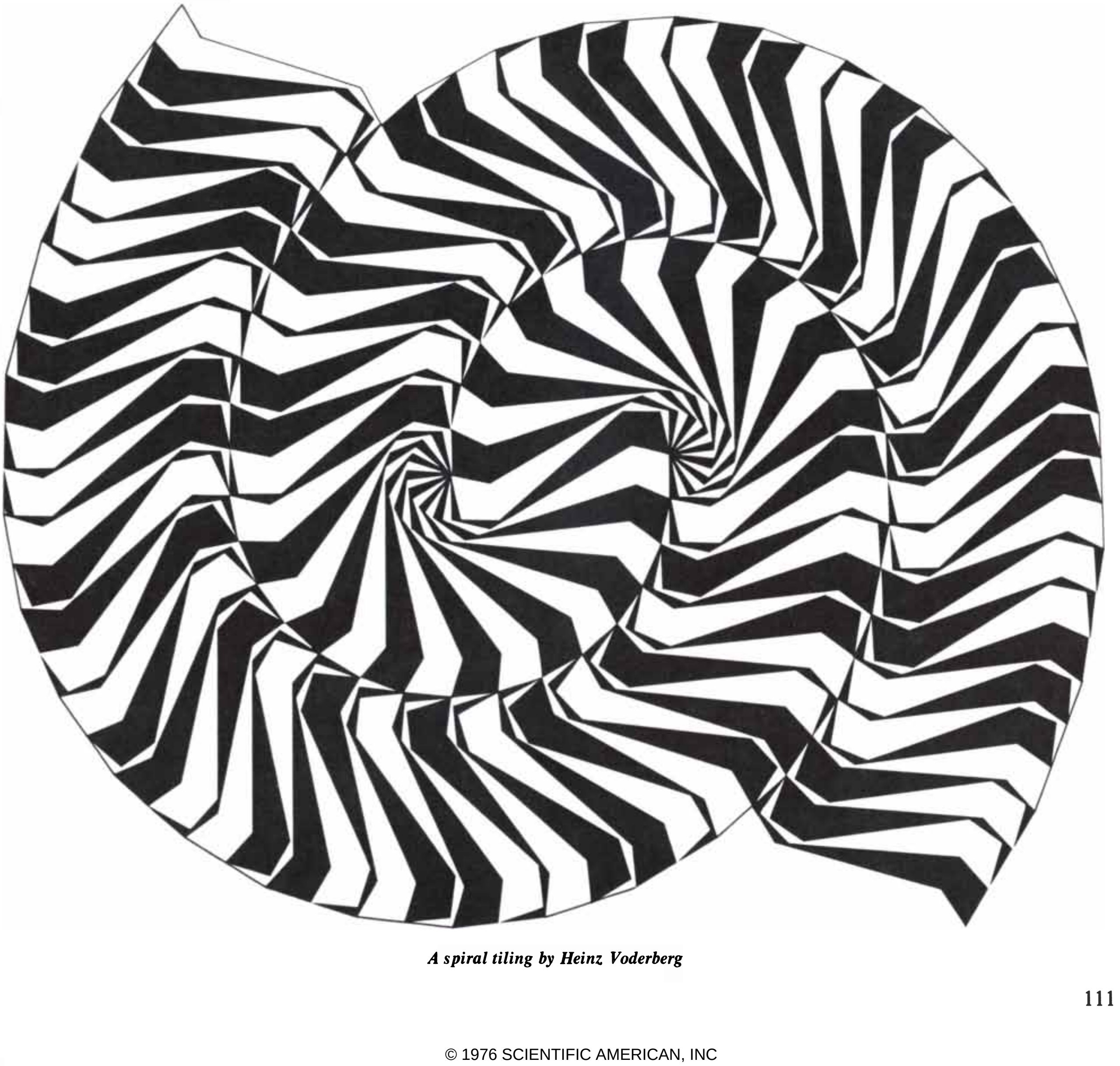
For many decades experts believed no such set exists, but the supposition proved to be untrue. In 1961 Hao Wang became interested in tiling the plane with sets of unit squares whose edges were colored in various ways. They are called Wang dominoes, and Wang wrote



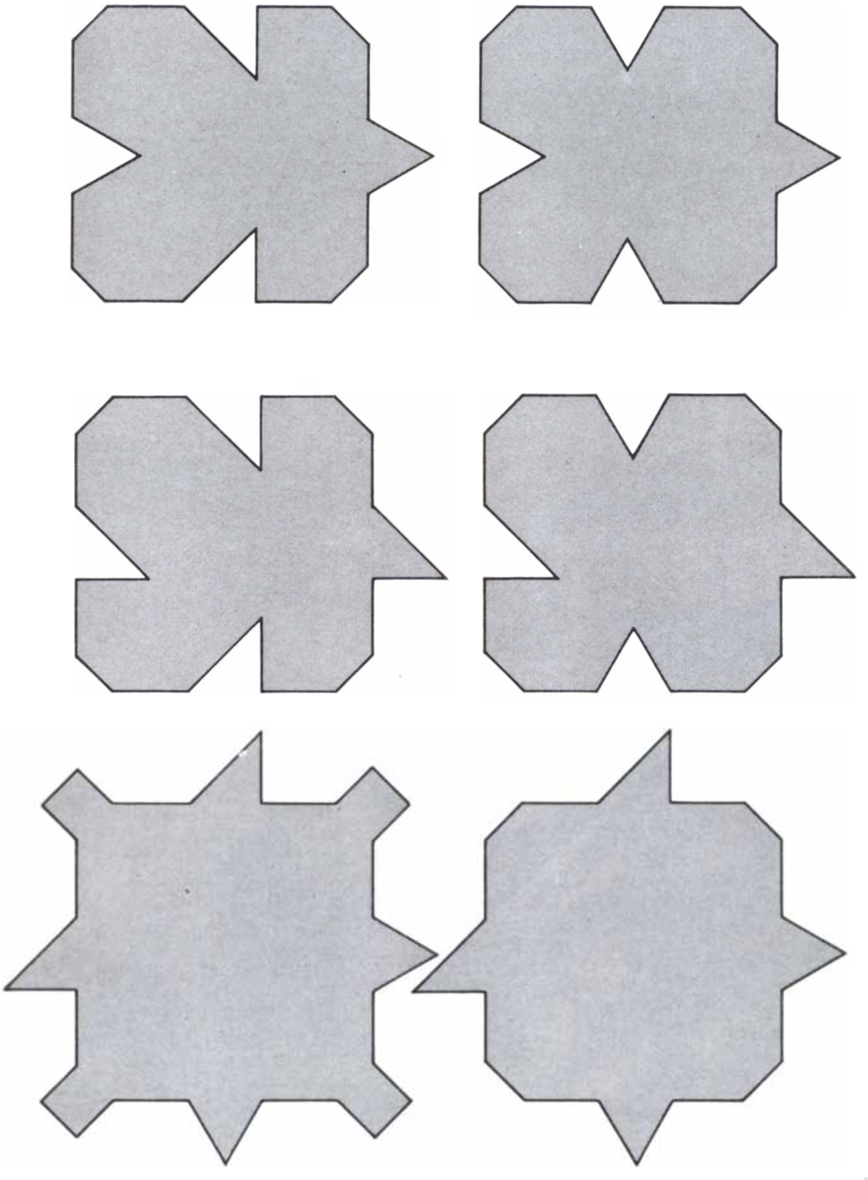
Nonperiodic tiling with congruent shapes



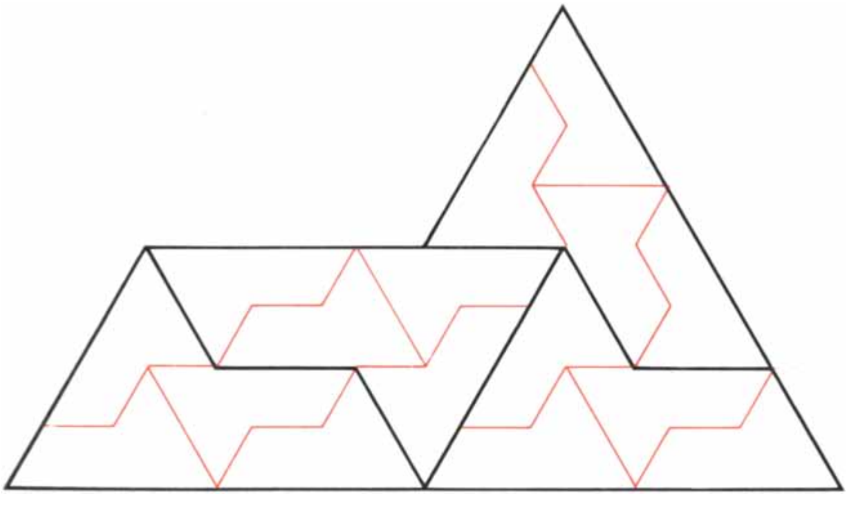
An enneagon (color at left) *and a pair of enneagons* (right) *forming an octagon that tiles periodically*



A spiral tiling by Heinz Voderberg



Raphael M. Robinson's six tiles that force a nonperiodic tiling



Three generations of sphinxes in a nonperiodic tiling

a splendid article about them for this magazine [see "Games, Logic and Computers," by Hao Wang; SCIENTIFIC AMERICAN, November, 1965]. Wang's problem was to find a procedure for deciding whether any given set of dominoes will tile by placing them so that abutting edges are the same color. Rotations and reflections are not allowed. The problem is important because it relates to decision questions in symbolic logic. Wang showed that if and only if there is a decision procedure, then any set of dominoes that tiles the plane nonperiodically will also tile periodically. He conjectured that such a procedure exists.

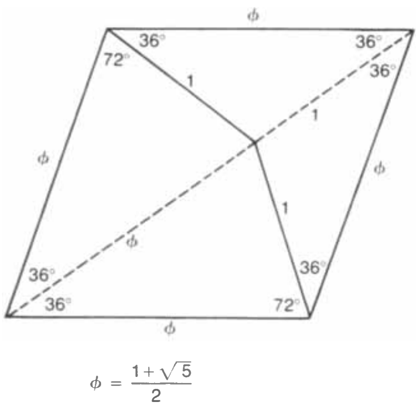
In 1964 Robert Berger, in his thesis for a doctorate from Harvard University in applied mathematics, showed that Wang's conjecture is false. There is no general procedure. Therefore there is a set of Wang dominoes that tiles only nonperiodically. Berger constructed such a set, using more than 20,000 dominoes. Later he found a much smaller set of 104. Last year Raphael M. Robinson reduced the set to 24.

It is easy to change such a set of Wang dominoes into polygonal tiles that tile only nonperiodically. You simply put projections and slots on the edges to make jigsaw pieces that fit in the manner formerly prescribed by colors. An edge formerly one color fits only another formerly the same color, and a similar relation obtains for the other colors. By allowing such tiles to rotate and reflect Robinson constructed six tiles [*see top illustration at left*] that "force nonperiodicity" in the sense explained above.

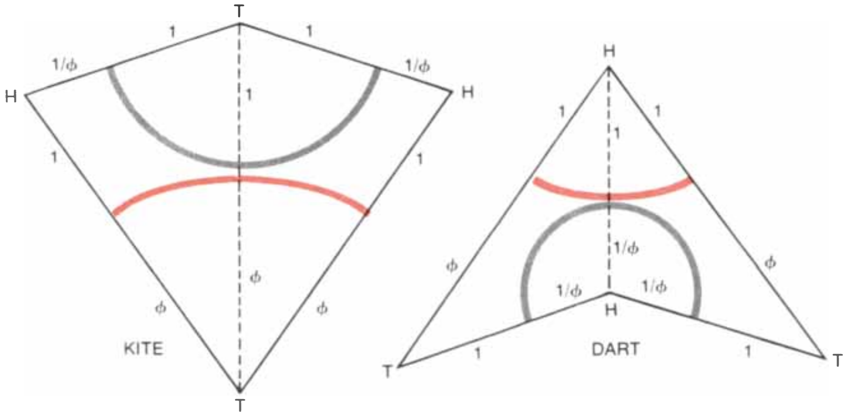
At the University of Oxford, where he is Rouse Ball Professor of Mathematics, Penrose searched for still smaller sets. Although most of his work is in relativity theory and quantum mechanics, he continues the active interest in recreational mathematics he shared with his geneticist father, the late L. S. Penrose. (They are the inventors of the famous "Penrose staircase" that goes round and round without getting higher; Escher depicted it in his lithograph "Ascending and Descending.") In 1973 Penrose found a set of six tiles that force nonperiodicity. Soon he found a way to reduce them to four, and in 1974 he lowered them to two.

Because the tiles lend themselves to commercial puzzles, Penrose was reluctant to disclose them until he had applied for patents in the United Kingdom, the U.S. and Japan. Now that these patents are pending, I have his permission to write about the tiles. I am equally indebted to John Horton Conway for many of the results of his study of the Penrose tiles.

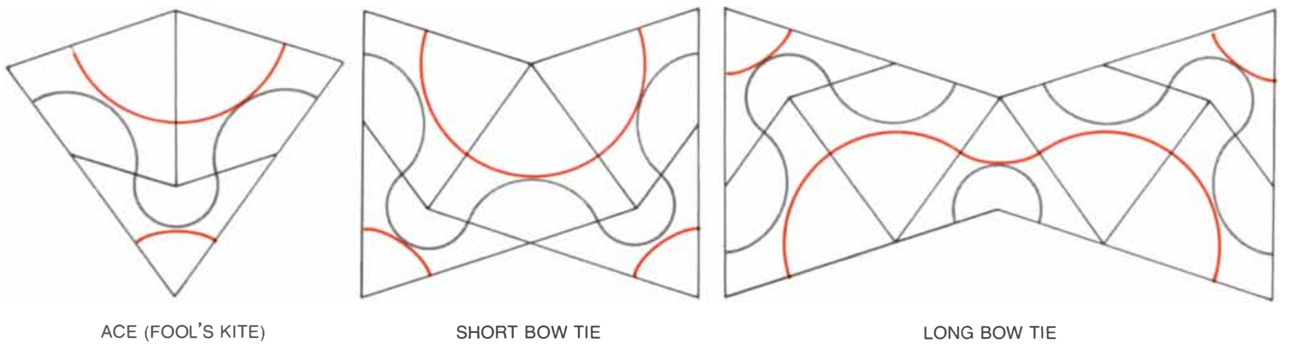
The shapes of a pair of Penrose tiles can vary, but the most interesting pair have shapes that Conway calls "darts"



Construction of dart and kite



A coloring of dart and kite to force nonperiodicity



Aces and bow ties that speed constructions

and "kites." The illustration at the top left on this page shows how they are derived from a rhombus with angles of 72 and 108 degrees. Divide the long diagonal in the familiar golden ratio of $(1 + \sqrt{5})/2 = 1.61803398\ldots$, then join the point to the obtuse corners. That is all. Let phi stand for the golden ratio. Each line segment is either 1 or phi as indicated.

The rhombus of course tiles periodically, but we are not allowed to join the pieces in this manner. Forbidden ways of joining sides of equal length can be enforced by bumps and dents, but there are simpler ways. For example, we can label the corners *H* and *T* (heads and tails) as is shown in the illustration at the top right on this page, and then give the rule that in fitting edges only corners of the same letter may meet. Dots of two colors could be placed in the corners to aid in conforming to this rule, but a prettier method, proposed by Conway, is to draw circular arcs of two colors on each tile, as the illustration shows. Each arc cuts the sides as well as the axis of symmetry in the golden ratio. Our rule is that abutting edges must join arcs of the same color.

To appreciate the full beauty and mystery of Penrose tiling one should make at least 100 kites and 60 darts. The pieces need be colored on one side only. The areas of the two shapes are in the golden ratio. This proportion also applies to the number of pieces you need of each type. You might think you need more of the smaller darts, but it is the other way around. You need 1.618... as many kites as darts. In an infinite tiling this proportion is exact.

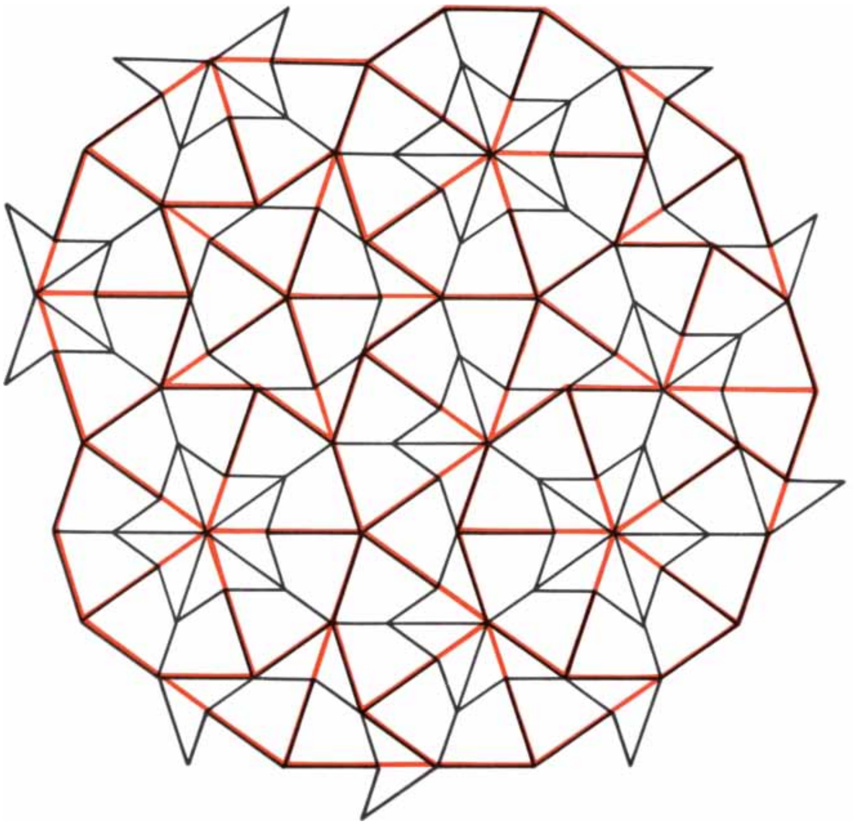
A good plan is to draw as many darts and kites as you can on one sheet, with a ratio of about five kites to three darts, using a thin line for the curves. The sheet can be photocopied many times. The curves can then be colored with, say, red and green felt-tip pens. Conway has found that it speeds constructions and keeps patterns stabler if you make many copies of the three larger shapes in the lower illustration on this page. As you expand a pattern you can continually replace darts and kites with aces and bow ties. Actually an infinity of arbitrarily large *pairs* of shapes, made up of darts and kites, will serve for tiling any infinite pattern.

A Penrose pattern is made by starting with darts and kites around one vertex and then expanding radially. Each time you add a piece to an edge you must choose between a dart and a kite. Sometimes the choice is forced, sometimes it is not. Sometimes either piece fits, but later you may encounter a contradiction (a spot where no piece can be legally added) and be forced to go back and make the other choice. It is a good plan to go around a boundary, placing all the forced pieces first. They cannot lead to a contradiction. You can then experiment with unforced pieces. It is always possible to continue forever. The more you play with the pieces, the more you will become aware of "forcing rules" that increase efficiency. For example, a dart forces two kites in its concavity, creating the ubiquitous ace.

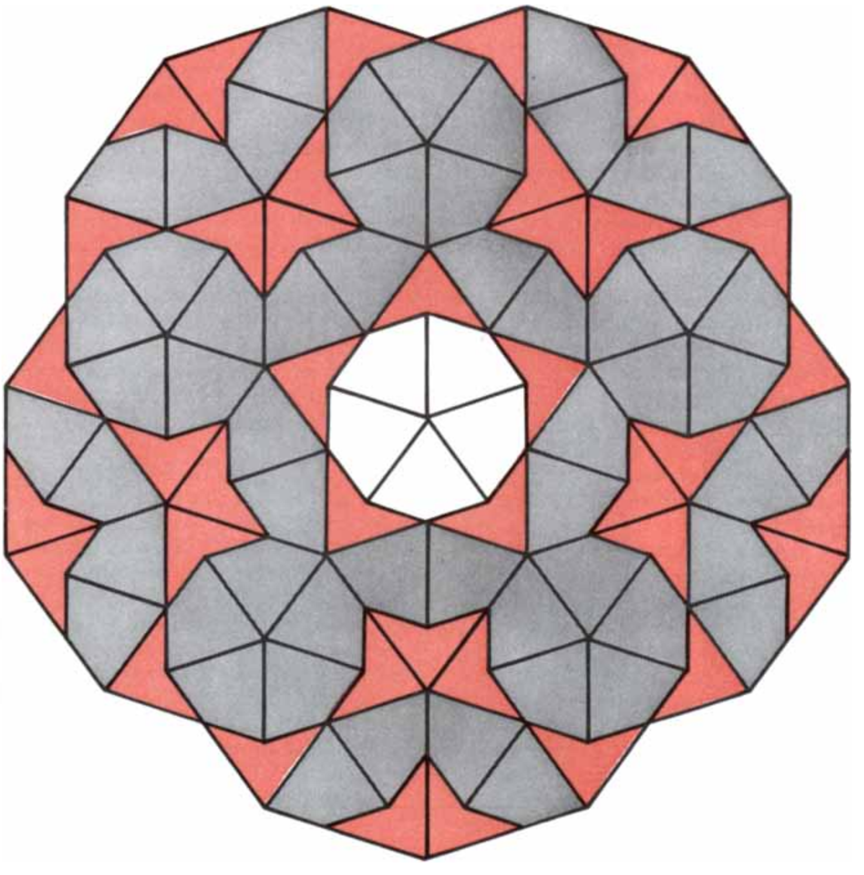
There are many ways to prove that the number of Penrose tilings is uncountable, just as the number of points on a line is. These proofs rest on a surprising phenomenon discovered by Penrose. Conway calls it "inflation" and "deflation." The top illustration on the next page shows the beginning of inflation. Imagine that every dart is cut in half and then all short edges of the original pieces are glued together. The result: a new tiling (shown in color) by larger darts and kites.

Inflation can be continued to infinity, with each new "generation" of pieces larger than the last. Note that the second-generation kite, although it is the same size and shape as a first-generation ace, is formed differently. For this reason the ace is also called a fool's kite. It should never be mistaken for a second-generation kite. Deflation is the same process carried the other way. On every Penrose tiling we can draw smaller and smaller generations of darts and kites. This pattern too goes to infinity.

Conway's proof of the uncountability of Penrose patterns (Penrose had earlier proved it in a different way) can be outlined as follows. On the kite label one side of the axis of symmetry *L*, the other



How a pattern is inflated



The infinite sun pattern

R (for left and right). Do the same on the dart, using *l* and *r*. Now pick a random point on the tiling. Record the letter that gives its location on the tile. Inflate the pattern one step, note the location of the same point in a second-generation tile and again record the letter. Continuing through higher inflations, you generate an infinite sequence of symbols that is a unique labeling of the original pattern seen, so to speak, from the selected point.

Pick another point on the original pattern. The procedure may give a sequence that starts differently, but it will reach a letter beyond which it agrees to infinity with the former sequence. If there is no such agreement beyond a certain point, the two sequences label distinct patterns. Not all possible sequences of the four symbols can be produced this way, but those that label different patterns can be shown to correspond in number with the number of points on a line.

We have omitted the colored curves on our pictures of tilings because they make it difficult to see the tiles. If you work with colored tiles, however, you will be struck by the beautiful designs created by these curves. Penrose and Conway independently proved that whenever a curve closes, it has a pentagonal symmetry, and the entire region within the curve has a fivefold symmetry. At the most a pattern can have two curves that do not close. In most patterns all curves close.

Although it is possible to construct Penrose patterns with a high degree of symmetry (an infinity of patterns have bilateral symmetry), most patterns, like the universe, are a mystifying mixture of order and unexpected deviations from order. As the patterns expand they seem to be always striving to repeat themselves but never quite managing it. G. K. Chesterton once suggested that an extraterrestrial being, observing how many features of a human body are duplicated on the left and the right, would reasonably deduce that we have a heart on each side. The world, he said, "looks just a little more mathematical and regular than it is; its exactitude is obvious, but its inexactitude is hidden; its wildness lies in wait." Everywhere there is a "silent swerving from accuracy by an inch that is the uncanny element in everything... a sort of secret treason in the universe." The passage is a nice description of Penrose's planar worlds.

There is something even more surprising about Penrose universes. In a curious finite sense, given by the "local isomorphism theorem," all Penrose patterns are alike. Penrose was able to show that every finite region in any pattern is contained somewhere inside every other pattern. Moreover, it appears infinitely many times in every pattern.

To understand how crazy this situation is, imagine that you are living on an infinite plane tessellated by one tiling of the uncountable infinity of Penrose tilings. You can examine your pattern, piece by piece, in ever expanding areas. No matter how much of it you explore you can never determine which tiling you are on. It is no help to travel far out and examine disconnected regions, because all the regions belong to one large finite region that is exactly duplicated infinitely many times on all patterns. Of course, this is trivially true of any periodic tessellation, but Penrose universes are not periodic. They differ from one another in infinitely many ways, and yet it is only at the unobtainable limit that one can be distinguished from another.

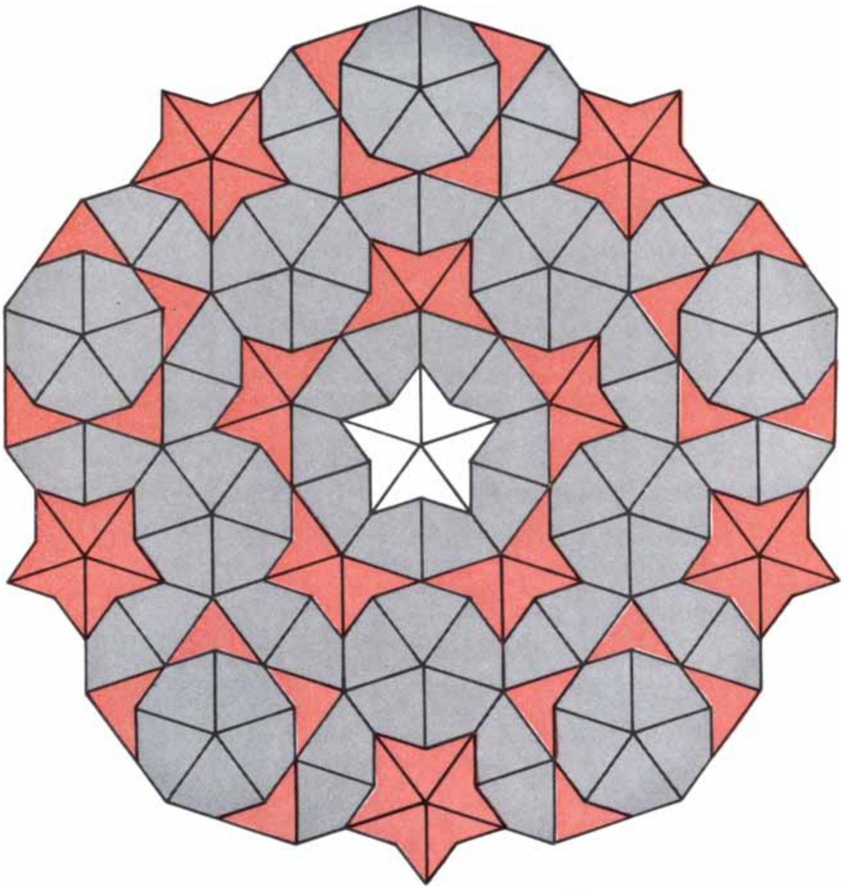
Suppose you have explored a circular region of diameter d. Call it the "town" where you live. Suddenly you are transported to a randomly chosen parallel Penrose world. How far are you from a region that exactly matches the streets of your home town? Conway answers with a truly remarkable theorem. The distance is never more than $2d$! (This is an upper bound, not an average.) If you walk in the right direction, you need not go more than a distance of $2d$ to find yourself inside an exact copy of your home town. The theorem also applies to the universe in which you live. Every large circular pattern (there is an infinity of different ones) can be reached by walking a distance in some direction that is certainly less than twice the diameter of the pattern and more likely about the same distance as the diameter.

The theorem is quite unexpected. Consider an analogous isomorphism exhibited by a sequence of unpatterned digits such as pi. If you pick a finite sequence of 10 digits and then start from a random spot in pi, you are pretty sure to encounter the same sequence if you move far enough along pi, but the distance you must go has no known upper bound, and the expected distance is enormously longer than 10 digits. The longer the finite sequence is, the farther you can expect to walk to find it again. On a Penrose pattern you are always very close to a duplicate of home.

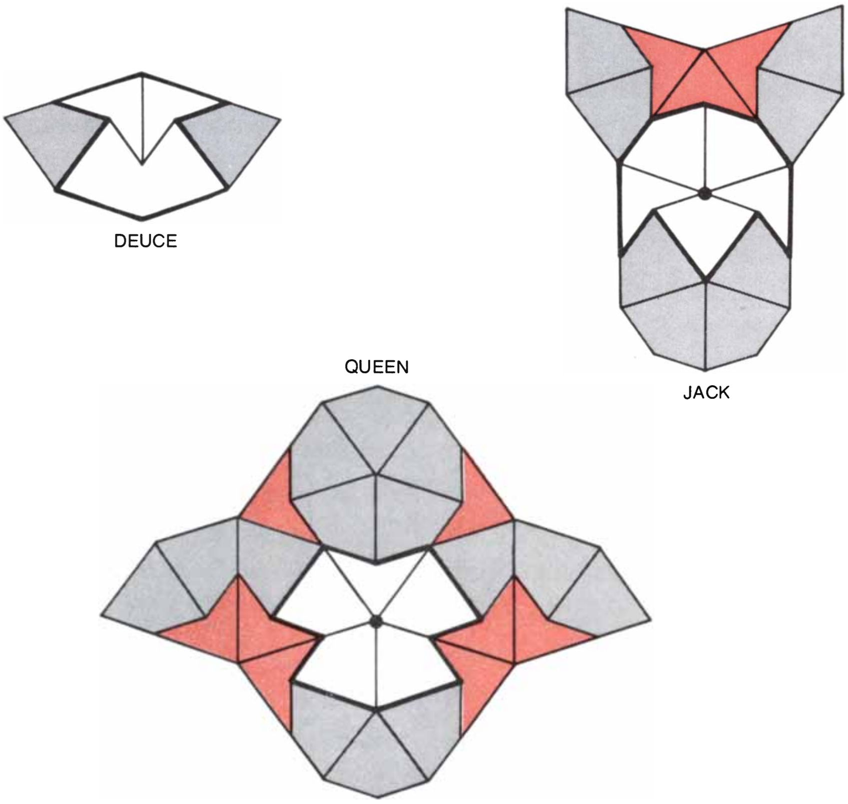
There are just seven ways that darts and kites will fit around a vertex. Let us consider first, using Conway's nomenclature, the two ways with pentagonal symmetry.

The sun (shown in white in the bottom illustration on the opposite page) does not force the placing of any other piece around it. If you add pieces so that pentagonal symmetry is always preserved, however, you will be forced to construct the beautiful pattern shown. It is uniquely determined to infinity.

The star, shown in white in the top illustration at the right, forces the 10 gray kites around it. Enlarge this pattern, always preserving the fivefold symmetry, and you will create another flowery design that is infinite and unique. The star and sun patterns are the only



The infinite star pattern



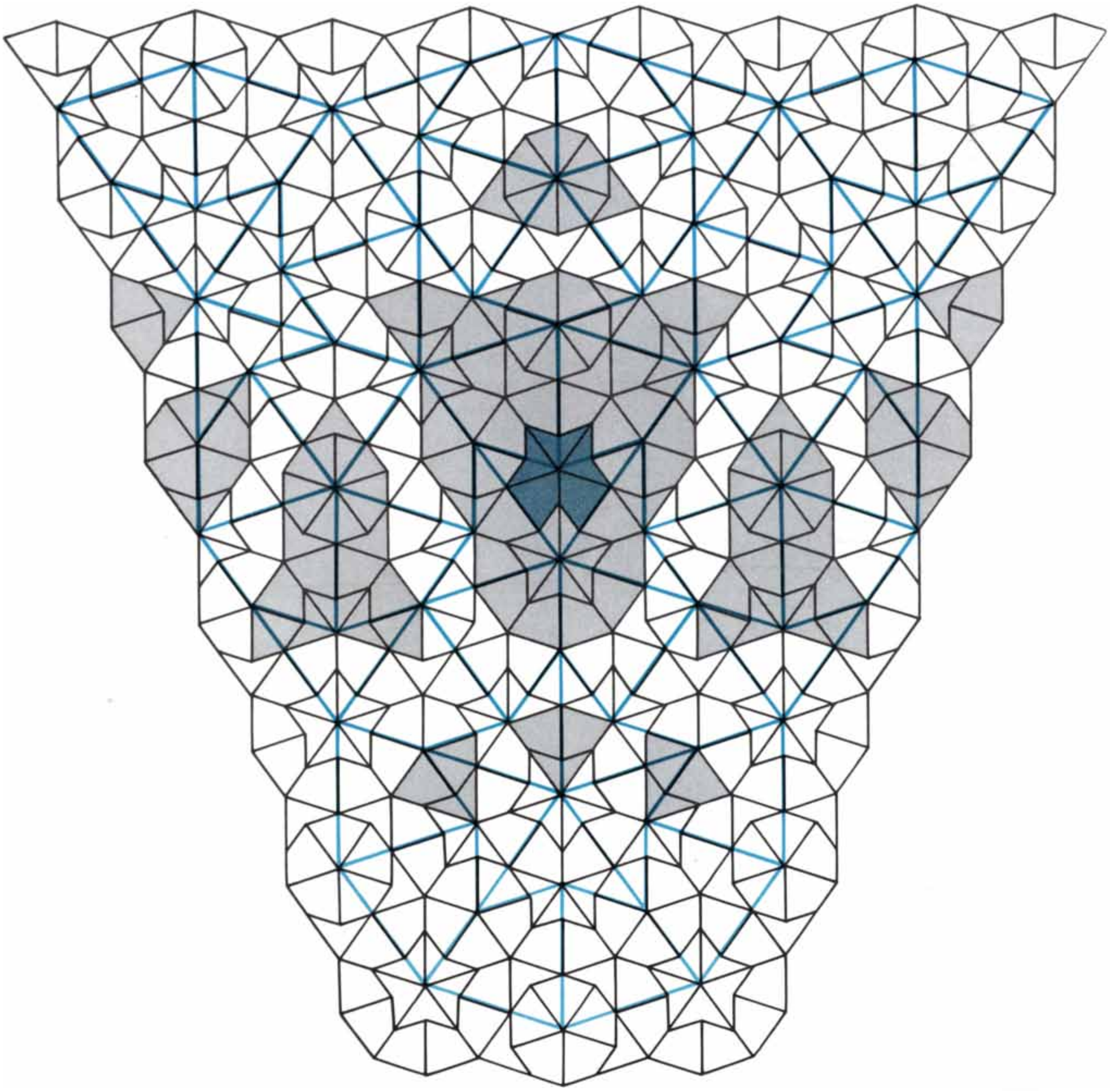
The "empires" of deuce, jack and queen

Penrose universes with perfect pentagonal symmetry, and there is a lovely sense in which they are equivalent. Inflate or deflate either of the patterns and you get the other.

The ace is a third way to tile around a vertex. It forces no more pieces. The deuce, the jack and the queen are shown in white in the bottom illustration on the preceding page, surrounded by the tiles they immediately force. As Penrose discovered (it was later found independently by Clive Bach), some of the seven vertex figures force the placing of tiles that are not joined to the immediately forced region. The illustration below shows in gray what is probably the major part of the king's "empire." (The king is the colored area.) All the gray tiles are forced by the king. (Two aces, just outside the left and right borders, are also forced but are not shown.)

This picture of the king's empire was drawn by a computer program written by Eric Regener of Concordia University in Montreal. His program deflates any Penrose pattern any number of steps. The colored lines show the domain immediately forced by the king. The black lines are a third-generation deflation in which the king and almost all of his empire are replicated. It is not known how much farther the empire extends, and the smaller empires of the jack and the queen have not been completely mapped.

The most extraordinary of all Penrose universes, essential for understanding the tiles, is the infinite cartwheel pattern, the center of which is shown in the illustration on the opposite page and on the cover. The regular decagon at the center, outlined in heavy black (each side is a pair of long and short edges), is what Conway calls a "cartwheel." Every point on any pattern is inside a cartwheel exactly like this one. By one-step inflation we see that every point will be



The king's empire

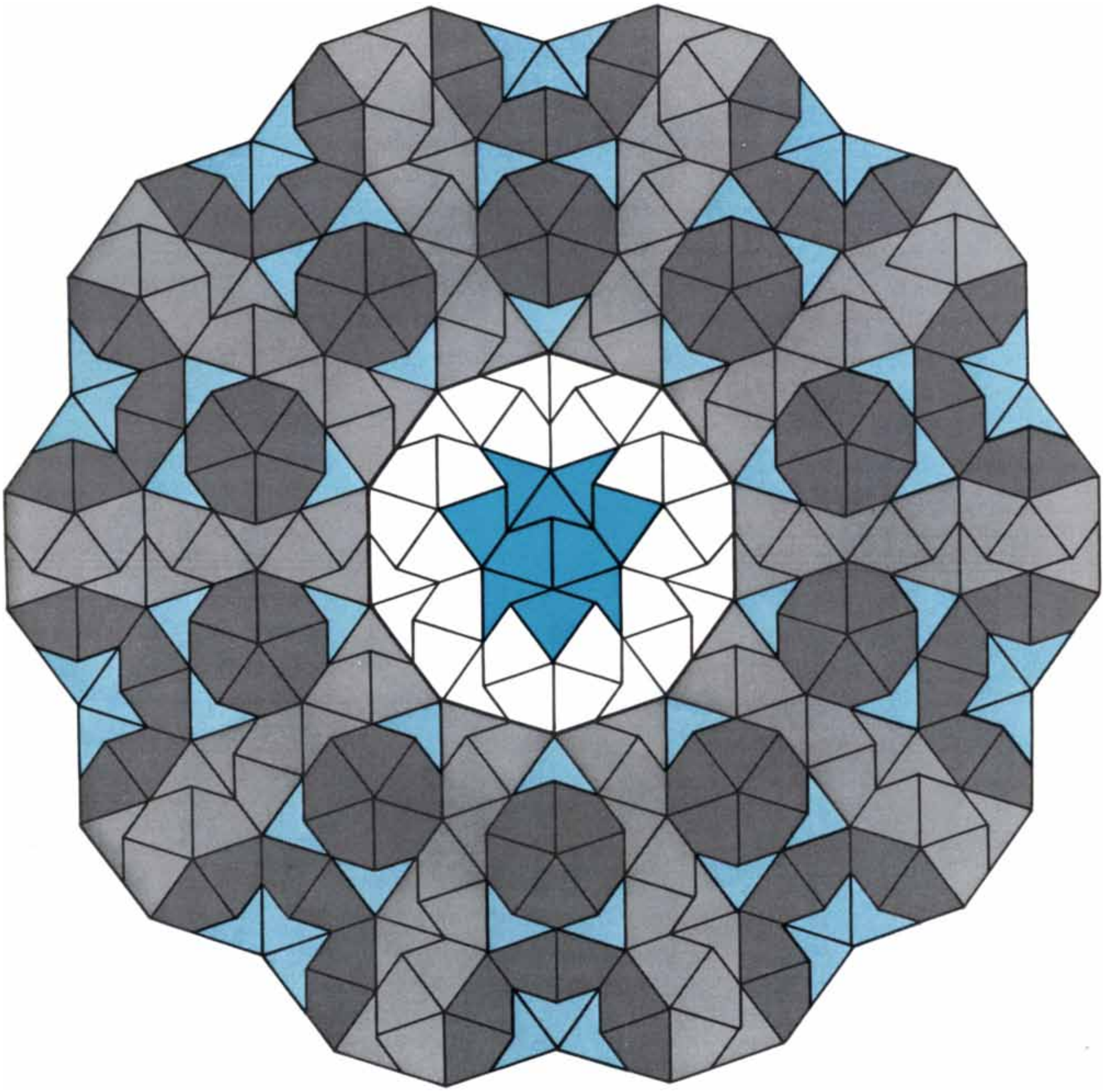
inside a larger cartwheel. Similarly, every point is inside a cartwheel of every generation, although the wheels need not be concentric.

Note the 10 light gray spokes that radiate to infinity. Conway calls them "worms." They are made of long and short bow ties, the long ones being in the golden ratio to the short ones. Every Penrose universe contains an infinite number of arbitrarily long worms. Inflate or deflate a worm and you get another worm along the same axis. Observe that two full worms extend across the central cartwheel in the infinite cartwheel pattern. (Inside it they are not gray.) The remaining spokes are half worms. Aside from spokes and the interior of the central cartwheel, the pattern has perfect tenfold symmetry. Between any two spokes we see an alternating display of increasingly large portions of the sun and star patterns.

Any spoke of the infinite cartwheel pattern can be turned side to side (or, what amounts to the same thing, each of its bow ties can be rotated end for end) and the spoke will still fit all surrounding tiles except for those inside the central cartwheel. There are 10 spokes; thus there are $2^{10} = 1{,}024$ combinations of states. After eliminating rotations and reflections, however, there are only 62 distinct combinations. Each combination leaves inside the cartwheel a region that Conway has named a "decapod."

Decapods are made up of 10 identical isosceles triangles with the shapes of half darts. The decapods with maximum symmetry are the buzzsaw and the starfish shown in the upper illustration on the next page. Like a worm, each triangle can be turned. As before, ignoring rotations and reflections, we get 62 decapods.

When the spokes are arranged the way they are in the infinite cartwheel pattern shown, a decapod called Bat-



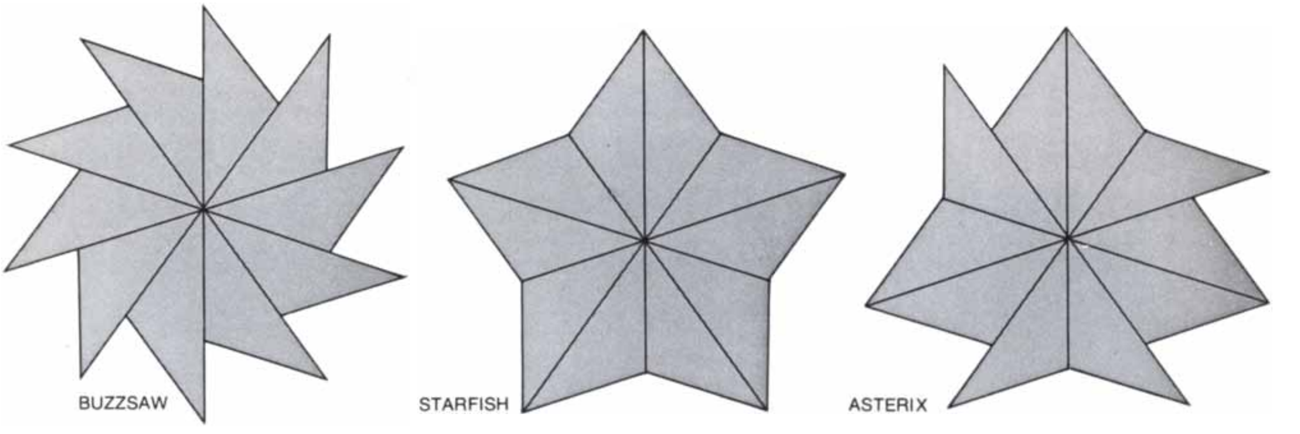
The cartwheel pattern

man is formed at the center. Batman (shown in dark color) is the only decapod that can legally be tiled. (No finite region can have more than one legal tiling.) Batman does not, however, force the infinite cartwheel pattern. It merely allows it. Indeed, no finite portion of a legal tiling can force an entire pattern, because the finite portion is contained in *every* tiling.

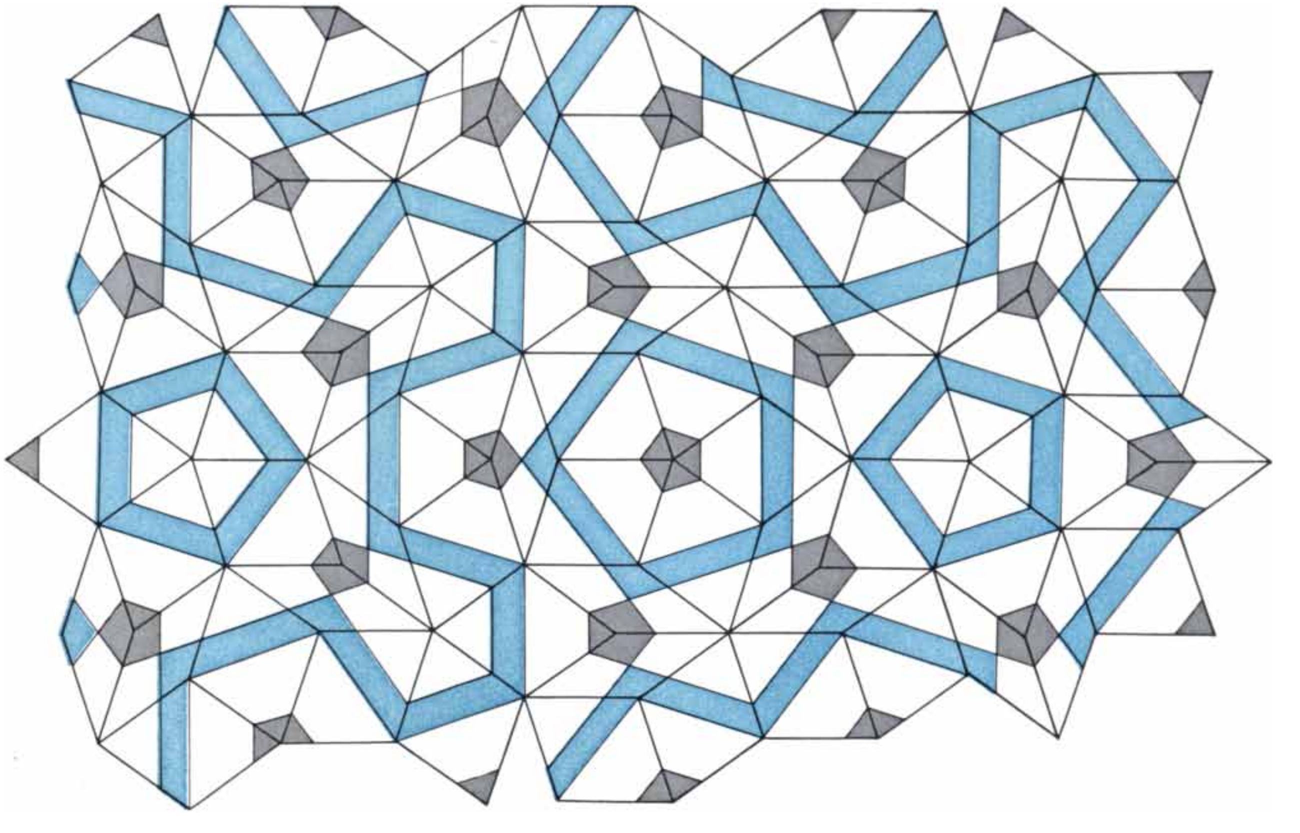
Note that the infinite cartwheel pattern is bilaterally symmetrical, its axis of symmetry going vertically through Batman. Inflate the pattern and it remains unchanged except for mirror reflection in a line perpendicular to the symmetry axis. The five darts in Batman and its two central kites are the only tiles in any Penrose universe that are not inside a region of fivefold symmetry. All other pieces in this pattern or any other one are in infinitely many regions of fivefold symmetry.

The other 61 decapods are produced inside the central cartwheel by the other 61 combinations of worm turns in the spokes. All are "holes" in the following sense. A hole is any finite empty region, surrounded by an infinite tiling, that cannot be legally tiled. You might suppose each decapod is the center of infinitely many tilings, but here Penrose's universes play another joke on us. Surprisingly, 60 decapods force a unique tiling that differs from the one shown only in the composition of the spokes. Only Batman and one other decapod, called Asterix after a French cartoon character, do not. Like Batman, Asterix allows an infinite cartwheel pattern, but it also allows patterns of other kinds.

Now for a startling conjecture. Conway believes, although he has not completed the proof, that every possible hole, of whatever size or shape, is equiv-



Three decapods



A nonperiodic tiling with Roger Penrose's rhombuses alent to a decapod hole in the following sense. By rearranging tiles around the hole, taking away or adding a finite number of pieces if necessary, you can transform every hole into a decapod. If this is true, any finite number of holes in a pattern can also be reduced to one decapod. We have only to remove enough tiles to join the holes into one big hole, then reduce the big hole until an untileable decapod results.

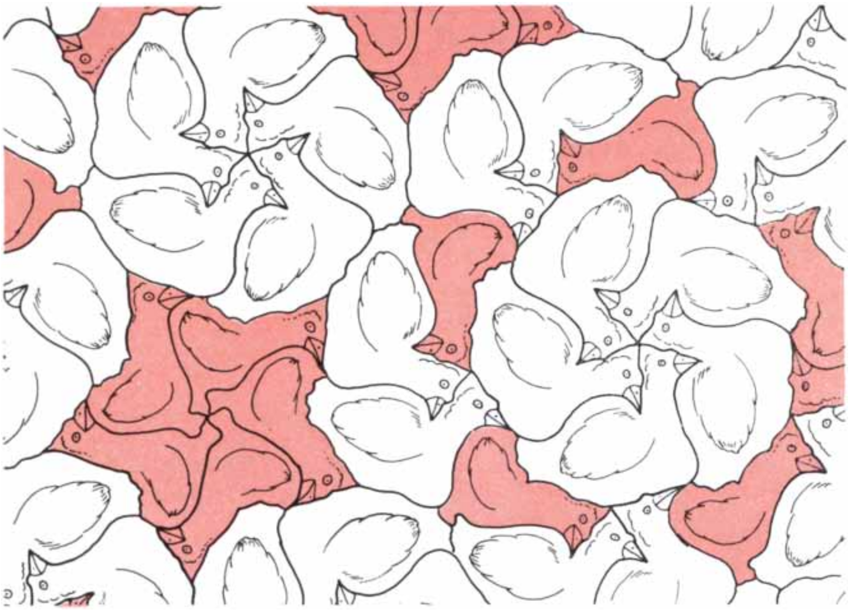
Think of a decapod as being a solid tile. Except for Batman and Asterix, each of the 62 decapods is like an imperfection that solidifies a crystal. It forces a unique infinite cartwheel pattern, spokes and all, that goes on forever. If Conway's conjecture holds, any "foreign piece" (Penrose's term) that forces a unique tiling, no matter how large the piece is, has an outline that transforms into one of 60 decapod holes.

Kites and darts can be changed to other shapes by the same technique described earlier for changing isosceles triangles into spiral-tiling polygons. It is the same technique that Escher employed for transforming polygonal tiles into animal shapes. The top illustration on this page shows how Penrose changed his darts and kites into chickens that tile only nonperiodically. Note that although the chickens are asymmetrical, it is never necessary to turn any of them over to tile the plane. Alas, Escher died before he could know of Penrose's tiles. How he would have reveled in their possibilities!

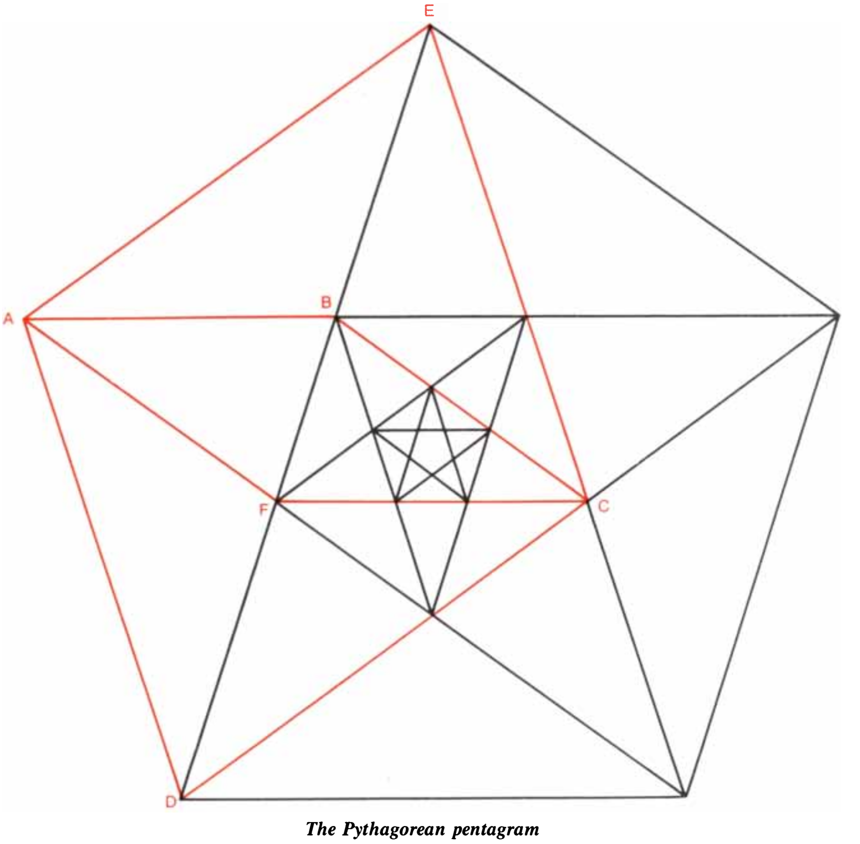
By dissecting darts and kites into smaller pieces and putting them together in other ways you can make other pairs of tiles with properties similar to those of darts and kites. Penrose found an unusually simple pair: the two rhombuses in the sample pattern in the bottom illustration on the opposite page. All edges are the same length. The larger piece has angles of 72 and 108 degrees and the smaller one has angles of 36 and 144 degrees. As before, both the areas and the number of pieces needed for each type are in the golden ratio. Tiling patterns inflate and deflate and tile the plane in an uncountable infinity of nonperiodic ways. The nonperiodicity can be forced by bumps and dents or by a coloring such as the one suggested by Penrose and shown in the illustration.

We see how closely the two sets of tiles are related to each other and to the golden ratio by examining the pentagram in the bottom illustration on this page. This was the mystic symbol of the ancient Greek Pythagorean brotherhood and the diagram with which Goethe's Faust trapped Mephistopheles. The construction can continue forever, outward and inward, and every line segment is in the golden ratio to the next smaller one. Note how all four Penrose tiles are embedded in the diagram. The kite is *ABCD,* the dart is *AECB.* The rhombuses, although they are not in the proper relative sizes, are *AECD* and *ABCF.* As Conway likes to put it, the two sets of tiles are based on the same underlying golden stuff.

Are there pairs of tiles not based on the golden ratio that force nonperiodic tiling? Is there a single piece that tiles only nonperiodically? These questions define two of the most intriguing problems that remain to be solved in the theory of tiling.



Penrose's nonperiodic chickens



The Pythagorean pentagram