Orbits of Orbs: Sphere Packing Meets Penrose Tilings

Charles Radin

1. INTRODUCTION. There is obvious value in finding the most efficient ways to arrange objects like balls or polyhedra in a given container. There is more scope for rich mathematics in having the "container" be all of space, so there are no boundaries to spoil the potential symmetry of the optimal arrangements, and this is the type of problem we will consider (see Figure 1 for an efficient and highly symmetric arrangement of unit circles in the plane). We emphasize that it is the *symmetry* of efficient arrangements that is our main concern.

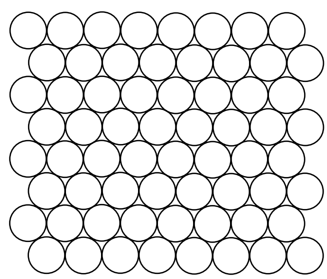


Figure 1. Part of a packing of disks in the Euclidean plane.

We will eventually quantify "efficiency" in terms of density, but first we establish a bit of more basic terminology. The standard term "packing" denotes an arbitrary arrangement, finite or infinite, of balls or polyhedra (usually members of some predetermined finite set of "model" objects) with pairwise disjoint interiors. The density of such a packing P in a bounded container is intuitively the sum of the volumes of the bodies (or portions thereof) in P divided by the volume of the container. One must somehow make a choice in order to generalize this intuitive notion when the container is all of space, and we make the usual choice by defining the density of a packing P as the limit, if it exists, of the fractional volume occupied by the bodies of P in a sphere as the sphere expands indefinitely about some common center (see Figure 2). There are subtleties involved in this approach that we will be forced to confront later, subtleties partly linked with our desire to include discussion of the analogous problem of densest packings of balls or polyhedra in hyperbolic space. We sidestep such issues for now, stating merely that our interest is in packings of optimal density.

2. PACKINGS IN EUCLIDEAN SPACE. We begin by concentrating on Euclidean spaces $\mathbb{E}^d$ (of arbitrary dimension d) and sharpen the discussion using the following definition.

Definition 1. By a *(densest) packing problem* we mean the following: given a finite collection $\mathcal{B}$ of bodies in $\mathbb{E}^d$, to find the densest packings of $\mathbb{E}^d$ by congruent copies of these bodies.

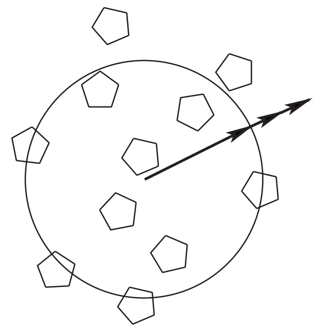


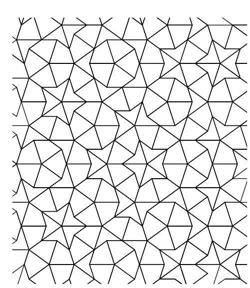
Figure 2. Expanding circle in a pentagonal packing.

In practice it is very difficult to solve densest packing problems. The main cases in which solutions exist are the following:

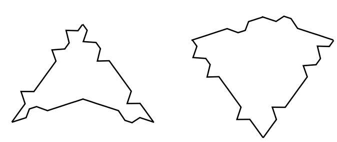
(a) $\mathcal{B}$ consists of a single regular n-gon in $\mathbb{E}^2$—but essentially just for $n = 3$, 4, and 6, namely, the cases for which the optimal density is 1 (tilings);

(b) $\mathcal{B}$ consists of a single disk in $\mathbb{E}^2$, and (presumably) $\mathcal{B}$ consists of a single ball in $\mathbb{E}^3$;

(c) so-called aperiodic tilings of $\mathbb{E}^d$, such as the kites & darts in $\mathbb{E}^2$ [**17**] (see Figure 3).



(a) A kite and dart tiling.



(b) The kite and dart tiles.

Figure 3.

The proof by Fejes Tóth that the hexagonal packing (Figure 1) is densest for congruent disks in $\mathbb{E}^2$ is beautiful and instructive [**16**]. It uses the notion of the "Voronoi region" of a body b in a packing P, which is the set of all points in $\mathbb{E}^d$ that are at least as close to b as to any other member of P. For every disk in the packing of Figure 1 the Voronoi region is a regular hexagon (see Figure 4). What Fejes Tóth showed is that, among all the Voronoi regions for any disk in any congruent disk packing, the relative fraction of the area of the Voronoi region that is filled by the disk is maximized by any of the disks in Figure 1. For the disks in that packing the relative fraction in question, which is $\pi/2\sqrt{3}$, is thus the highest possible density of any packing by congruent disks in $\mathbb{E}^2$: intuitively, the density has been maximized simultaneously in all local regions. The arguments by Hsiang [**20**] and by Hales [**18**] for ball packings in $\mathbb{E}^3$ share this approach of simultaneous local optima, though the quantity being optimized is no longer simply the relative volume of the Voronoi region (see [**22**]). And we note that the other solved problems listed, being tilings, share this feature of simultaneous local optimization.

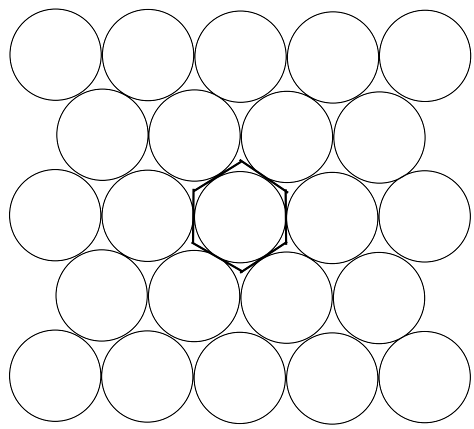


Figure 4. A Voronoi region in the packing of Figure 1.

Although densest packing problems are at the heart of our discussion, we will not be directly concerned with the actual solution of such problems, but with broader questions such as the existence, and especially the qualitative features (symmetries) of their solutions. Our rationale is that, although there are often significant applications for the solution of a densest packing problem, especially for ball packings, the deeper interest for mathematics lies in the ideas that the investigation of such problems spawns, particularly concerning subgroups of the isometry group $\mathcal{G}^d$ of $\mathbb{E}^d$ (or of d-dimensional hyperbolic space $\mathbb{H}^d$). The richness of this approach was recognized many years ago, for instance by Hilbert in his eighteenth problem, "Building Up of Space from Congruent Polyhedra" [**19**, pp. 466–467], in which he posed three specific questions that we paraphrase as follows:

(i) Are there only finitely many possible symmetry groups of densest packings of bodies in $\mathbb{E}^d$? (It was already known in Hilbert's time that there are infinitely many in $\mathbb{H}^d$.)

(ii) Is it the case for each polyhedron that can tile $\mathbb{E}^d$ or $\mathbb{H}^d$ that there is (also) a tiling consisting precisely of the images of the same polyhedron under the action of a subgroup of the group of rigid motions of $\mathbb{E}^d$ or $\mathbb{H}^d$?

(iii) What are the different structures (in particular, symmetries) of the densest packings of $\mathbb{E}^d$ or $\mathbb{H}^d$?

The history of research on Hilbert's eighteenth problem is erratic. Bieberbach gave a celebrated solution of (i) in 1910 [**2**]. However, a simple counterexample to (ii) was given in 1928 by Reinhardt [**32**], and no significant progress was ever made on (iii) (in the context of general $\mathbb{E}^d$ or $\mathbb{H}^d$). Given this discouraging history, the problem has largely passed from view.

There is a prescient note in Milnor's review of Hilbert's eighteenth problem [**24**] concerning the seminal paper by Robinson on aperiodic tilings [**33**], in which Milnor shows their connection with problem (ii). For us their connection with problem (iii) is more useful, as an indication of how symmetry may be manifested in packing problems. In fact this article could be considered an analysis of the impact of the study of aperiodicity on Hilbert's problem.

Packing problems with unique solution. Consider again the simplest interesting solved example of a packing problem, the densest packings of $\mathbb{E}^2$ by congruent disks. We noted earlier that the hexagonal packing is densest (by any reasonable definition), but we might inquire whether there are other packings, not congruent to this one, that have the same density. (This has been considered, for many years, a minor difficulty by those analyzing densest packings; see [**10**].) Clearly we could remove a finite number of disks from the hexagonal packing without affecting the density. Furthermore, we could actually remove certain infinite subcollections, such as the disks with centers on a line, without effect on the density, or we could (carefully) separate the disks so that no two are touching, and leave the density unchanged. However, using a recently modified definition of densest packing (which, as will be discussed later in the article, adds a criterion that the packing be suitably "nice"), it has been shown that the hexagonal packing is indeed the unique densest packing for unit disks—unique up to rigid motion of course [**7**]. This is significant since, being chiefly interested in understanding the symmetries of densest packings, we could hardly proceed if, for examples such as congruent disks in the plane, we could not effectively distinguish between a given packing and one obtained from it by removing a few bodies, thereby destroying symmetries.

Some packing problems do not show the uniqueness exhibited by unit disk packings. The problem for unit balls in $\mathbb{E}^3$ does not, since it includes among its solutions the face centered cubic packing and the hexagonal close packed structure. To analyze this situation further, we first recall a recent result, which we prove later, connecting uniqueness and symmetry [**7**, Theorem 2].

Theorem 1. *If a densest packing problem has a solution P that is unique up to rigid motion, then the symmetry group of P must be cocompact in the Euclidean group.*

We remind the reader that a subgroup H of a topological group G is cocompact (respectively, cofinite) if the left coset space G/H is compact in the quotient topology (respectively, of finite volume for the quotient of Haar measure). (In the case of the isometry group $\mathcal{G}^d$ of $\mathbb{E}^d$ or $\mathbb{H}^d$ a discrete subgroup is cocompact if and only if one—and therefore all—of its fundamental domains is compact [**21**].)

The packing problem for kites & darts (see Figure 3) also fails to exhibit the form of uniqueness shown by the problem for unit disks. In fact we note here two features of the kite & dart tilings: there are uncountably many pairwise noncongruent tilings of this type; these tilings form a natural family, in the sense that any finite region of one kite & dart tiling has a congruent copy in any other kite & dart tiling. In other words, although there is not, modulo rigid motion, just one densest packing (tiling) of $\mathbb{E}^2$ by kites & darts, the situation is in an intuitive sense not as bad as the problem for unit balls in $\mathbb{E}^3$, where (presumably) the optimal packings are not all locally congruent. The packing problem for kites & darts is called aperiodic since *none* of its densest packings—in this case tilings—has a cocompact symmetry group.

We will be mainly concerned with the symmetries of densest packings, and it will be convenient to translate the traditional classification of symmetry groups into other terms. But first we need the following basic notion of conjugacy of group actions.

Definition 2. Actions π_1 and π_2 of a group G on spaces X_1 and X_2, respectively, are *conjugate* if there is a bijection ϕ of X_1 onto X_2 such that

$$\phi\big[\pi_1(g)x\big] = \pi_2(g)\phi[x]$$

for all x in X_1 and g in G.

The following fact will prove useful.

Theorem 2. *Subgroups H_1 and H_2 of a group G are inner isomorphic (often called conjugate) if and only if the left actions π_1 and π_2 of G on the coset spaces G/H_1 and G/H_2, respectively, are conjugate in the sense of Definition 2.*

Proof. Suppose that $\phi : G/H_1 \mapsto G/H_2$ is a conjugacy. Fix an element k of $\phi([e]_1)$, where e is the identity of G and $[\cdot]_j$ denotes an element of G/H_j, i.e., $[g]_j = gH_j$. Then

$$\phi\big([g]_1\big) = \phi\big([ge]_1\big) = \phi\big(g[e]_1\big) = g\phi\big([e]_1\big) = g[k]_2 = [gk]_2.$$

Also,

$$g \in H_1 \Leftrightarrow [g]_1 = [e]_1 \Leftrightarrow [gk]_2 = [k]_2 \Leftrightarrow (gk)^{-1}k \in H_2 \Leftrightarrow k^{-1}g^{-1}k \in H_2.$$

It follows that $k^{-1}H_1k = H_2$, so H_1 and H_2 are conjugate. The other direction is trivial. ■

Consider a packing P of $\mathbb{E}^d$ with cocompact symmetry group Γ_P and the action π of the Euclidean group $\mathcal{G}^d$ on the left quotient space $\mathcal{G}^d/\Gamma_P$: $\pi(g)[h] = [gh]$, where $[h]$ signifies the equivalence class $h\Gamma_P$ of h in $\mathcal{G}^d/\Gamma_P$. Theorems 1 and 2 show that, among those packing problems with unique solution, the classification of densest packing problems by the symmetry groups Γ_P of their solutions P is *equivalent* to their classification by the conjugacy classes of the actions of $\mathcal{G}^d$ on the coset spaces $\mathcal{G}^d/\Gamma_P$.

This can be made a little neater by bringing into play the natural bijection between $\mathcal{G}^d/\Gamma_P$ and the orbit $\mathcal{O}(P)$ of the packing P under the natural action of $\mathcal{G}^d$, namely, $g\Gamma_P \leftrightarrow gP$ for g in $\mathcal{G}^d$, which is well-defined since $g\Gamma_P = g'\Gamma_P$ if and only if $g^{-1}g'$ belongs to Γ_P, which is true precisely when $gP = g'P$. In fact, it is easy to see that this bijection provides a conjugacy.

Combining this observation with our previous use of Theorems 1 and 2, we see that the classification of those packing problems whose solutions are unique up to rigid motion in terms of the inner isomorphism classes of the (cocompact) symmetry groups Γ_P of their (unique) solutions P is *equivalent* to the classification of those packing problems by the conjugacy classes of the actions of $\mathcal{G}^d$ on the orbits $\mathcal{O}(P)$ of the solutions P.

Packings without unique solutions. Having completed our discussion of packing problems with unique solution (modulo rigid motion), we turn to packing problems whose solutions need not be unique. The first step is to generalize to these problems the classification scheme we applied to the simpler problems, and we choose to generalize the dynamical version. We begin by letting $\mathcal{P}_{\mathcal{B}}$ be the set of all possible packings corresponding to a given set $\mathcal{B}$ of bodies and defining a metric topology on $\mathcal{P}_{\mathcal{B}}$ such that a sequence of packings P_j converges to P in the topology if it converges uniformly on compact subsets of $\mathbb{E}^d$, where the latter convergence is in the Hausdorff metric. (It is not hard to write an explicit formula for a metric on $\mathcal{P}_{\mathcal{B}}$ whose associated topology is the one in question, but the metric is neither canonical nor particularly illuminating; the curious reader may find a description of it in [**28**].) Endowed with this structure, $\mathcal{P}_{\mathcal{B}}$ becomes a compact space. (If this seems surprising, think of the product topology on an infinite product of two-point sets.) We denote by $\tilde{\mathcal{P}}_{\mathcal{B}}$ the closure in $\mathcal{P}_{\mathcal{B}}$ of the set of densest packings. For this general situation we seek to classify up to topological conjugacy the dynamical systems that are represented by the natural action of $\mathcal{G}^d$ on $\tilde{\mathcal{P}}_{\mathcal{B}}$ for different $\mathcal{B}$. (A conjugacy is a *topological conjugacy* if the spaces involved are topological spaces and the bijection is a homeomorphism.)

Effectively we are defining the *symmetry of the packing problem* associated with $\mathcal{B}$ as the conjugacy class of the dynamical system of $\mathcal{G}^d$ acting on $\tilde{\mathcal{P}}_{\mathcal{B}}$. We reiterate that for the special subclass of packing problems with unique solutions the complete set of invariants reduces to the set of inner isomorphism classes of cocompact subgroups of $\mathcal{G}^d$, the possible symmetry groups of the solutions.

A good example of the need for a more general notion of symmetry for packing problems with nonunique solution is furnished by the case of the kite & dart tilings. These tilings attracted great attention in the mid '80s when physicists used three-dimensional analogues of the kite and dart tiles to model materials that exhibited unexpected behavior. These "quasicrystals" generated diffraction patterns with symmetries that were incompatible with the expected internal structure of the materials. Eventually it was shown that the diffraction pattern was directly related to the dynamical system associated with the tilings of the model [**14**], and more specifically, that the rotational symmetry of the diffraction patterns could be precisely understood as the rotational symmetry of all translation-invariant measures on the space of tilings [**28**]. In other words, for these tilings rotational symmetry exhibited itself not as symmetries of the tilings per se, but as *symmetries of measures on the space of tilings*. (Intuitively, lifting the operation of symmetries from packings to measures on packings is similar to the employment of weak solutions of differential equations, as is necessary in some situations, for instance in quantum field theories.)

As is the case for kite & dart tilings, in practice one obtains invariants for packing problems with nonunique solutions using subgroups of $\mathcal{G}^d$ that are *not cocompact*. Another example arose in the classification of certain tilings in $\mathbb{E}^3$ [**11**], [**29**], which led to the analysis of the family of (typically dense) subgroups of SO(3) generated by pairs of finite order rotations about axes that meet at a "geodetic" angle, that is, an angle α for which $\sin^2(\alpha)$ is rational. A series of papers ensued in which such groups were almost completely classified, with unexpected consequences [**30**], [**31**], [**12**], [**13**]. This is the sort of result one expects from such a program: the search for conjugacy invariants of dynamical systems leads to the investigation of certain non-cocompact subgroups of $\mathcal{G}^d$, which exposes information about them that is of independent interest. Other examples will be given later in the paper when we discuss packings in hyperbolic spaces.

It is now time to give the full technical definition of densest packings.

Definition 3. A packing P that lies in the support of a $\mathcal{G}^d$-invariant Borel probability measure μ on $\mathcal{P}_{\mathcal{B}}$ is *generic* for μ if

$$\int_{\mathcal{P}_{\mathcal{B}}} f(P')\, d\mu(P') = \lim_{n\to\infty} \frac{\int_{\mathcal{G}_n^d} f(gP)\, d\nu(g)}{\int_{\mathcal{G}_n^d} 1\, d\nu(g)}$$

holds for every continuous function f on $\mathcal{P}_{\mathcal{B}}$, where $\mathcal{G}_n^d = \{g \in \mathcal{G}^d \mid d(O, gO) \leq n\}$, O is an origin in $\mathbb{E}^d$, and ν is Haar measure on $\mathcal{G}^d$. (This definition is independent of the choice of O.)

Intuitively, when a group G acts on a compact metric space X that carries a measure μ invariant under the action, a point x in X is generic with respect to μ if, for any given point x' in X and ball $B(x')$ centered at x', the fraction of the orbit of x under the action of G that lies in $B(x')$ is $\mu[B(x')]$. The main fact about generic points is the Pointwise Ergodic Theorem of G.D. Birkhoff, which establishes the existence of—indeed, the abundance of—such generic points [**34**].

Theorem 3 (Birkhoff). *If $\mathcal{G}^d$ acts continuously on a compact metric space X and if a probability measure μ on X is ergodic under this action, then the set of generic points for μ has full μ-measure.*

Suppose that a group G acts on a space X. Then a probability measure μ on X that is invariant under the action of G is "ergodic" if the only subsets A of X that are invariant under the action of G satisfy $\mu(A) = 0$ or $\mu(A) = 1$.

We first use this basic theorem to foliate $\mathcal{P}_{\mathcal{B}}$ [**7**, Lemma 1].

Theorem 4. *If a probability measure μ on $\mathcal{P}_{\mathcal{B}}$ is ergodic under $\mathcal{G}^d$, then for each generic packing P of μ and each point O in $\mathbb{E}^d$ the "density"*

$$\lim_{n\to\infty} \frac{\mathrm{vol}[P \cap B_n(O)]}{\mathrm{vol}[B_n(O)]}$$

exists and has value $\mu(A_{\mathcal{B},O})$, where $B_n(O)$ is the ball of radius n centered at O and $A_{\mathcal{B},O}$ is the set $\{P \in \mathcal{P}_{\mathcal{B}} \mid O$ lies in some member of $P\}$.

(Note that $\mu(A_{\mathcal{B},O})$ is independent of O since μ is invariant.) This shows that each ergodic measure μ singles out packings of a given fixed density, thus foliating $\mathcal{P}_{\mathcal{B}}$ by density. It is now natural to define densest packings as follows.

Definition 4. A member P of $\mathcal{P}_{\mathcal{B}}$ is a *densest packing* if it is generic for some ergodic measure μ such that $\mu(A_{\mathcal{B},O}) = \sup_{\mu'} \mu'(A_{\mathcal{B},O})$, where the supremum extends over all probability measures μ' on $\mathcal{P}_{\mathcal{B}}$ that are ergodic with respect to the $\mathcal{G}^d$-action.

The fundamental question of the existence of densest packings now follows rather easily: it can be proven that for any $\mathcal{B}$ there exists on $\mathcal{P}_{\mathcal{B}}$ a (not necessarily unique) probability measure μ that is ergodic for the $\mathcal{G}^d$ action and satisfies $\mu(A_{\mathcal{B},O}) = \sup_{\mu'} \mu'(A_{\mathcal{B},O})$ [**8**]. Also, using Definition 4 for densest packing we can prove, as stated earlier, that the packing problem for congruent disks in $\mathbb{E}^2$ has a solution that is unique up to rigid motion [**8**].

The proof of Theorem 1 also now follows easily. By assumption there is an invariant probability measure μ on the orbit $\mathcal{O}(P)$ of the unique solution P. But, as shown earlier, $\mathcal{O}(P)$ can be identified with $\mathcal{G}^d/\Gamma_P$, so by the uniqueness of Haar measure μ must be the projection of Haar measure from $\mathcal{G}^d$ to $\mathcal{G}^d/\Gamma_P$, that is, Γ_P is cofinite. In the Euclildean setting this implies that Γ_P is cocompact.

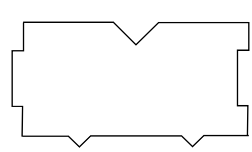
3. PACKING IN HYPERBOLIC SPACE $\mathbb{H}^d$.

We begin with a short introduction to the Poincaré upper half-plane model of $\mathbb{H}^2$. The space is identified with the subset $H = \{z = x + iy \mid y > 0\}$ of the complex plane, with the metric $ds^2 = (dx^2 + dy^2)/y^2$ and area $dA = dx\,dy/y^2$. It is easy to show that the geodesics are the semicircular arcs in H that approach the x-axis perpendicularly, including half-lines orthogonal to the x-axis as special cases. The rigid motions, meaning those motions continuously connected to the identity and preserving the metric, are identifiable with the matrix group $\mathcal{G}^2 = \mathrm{PSL}(2, \mathbb{R})$, acting on H by fractional linear transformation:

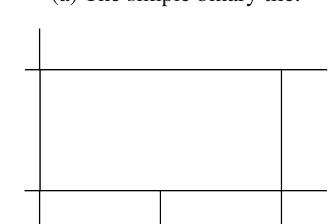
$$z \mapsto \frac{az + b}{cz + d},$$

where a, b, c, and d are real and $ad - bc = 1$. (The group $\mathrm{PSL}(2, \mathbb{R})$ is obtained from the matrix group $\mathrm{SL}(2, \mathbb{R})$, which comprises all 2×2 real matrices of determinant one, by identifying the transformation corresponding to a, b, c, d with that corresponding to $-a$, $-b$, $-c$, $-d$.) Special cases are the horizontal translation $z \mapsto z + b$ for b in $\mathbb{R}$ and the dilation $z \mapsto mz$ for m in $\mathbb{R}_+$.

A very interesting example of a packing problem in this context was considered by Penrose in 1978 [**27**], with $\mathcal{B}$ consisting of a single body, the "binary tile" shown in Figure 5a. Starting with this body horizontal motions fill out a strip, and then dilation by 2^j for j in $\mathbb{Z}$ yields a tiling of the hyperbolic plane (Figure 5b). It is not the only tiling by copies of the binary tile. Once a tile is placed in the plane, for instance as in Figure 5a, a "horozontal" strip is forced. We need here the notion of *horoball*, a region in H that is either inside a Euclidean circle tangent to the x-axis or above a Euclidean line parallel to the x-axis. The boundary of a horoball is called a *horocycle*. The two horizontal edges of the strip referred to earlier are horocycles, and the tiling of the region outside ("below") the larger of the two horoballs is also forced. But there are (uncountably) many ways to fill the horoballs. Penrose gave a beautiful argument showing that no such tiling could have cocompact symmetry group. If it did, it would tile a compact fundamental domain for the group $\mathcal{G}^2$, necessarily using only finitely many tiles to do this. But each tile has more outgoing bumps than ingoing bumps, and this "imbalance" produces a contradiction. (This has been nicely generalized in [**3**] and [**23**].)



(a) The simple binary tile.



(b) The binary tiling of the hyperbolic plane.

Figure 5.

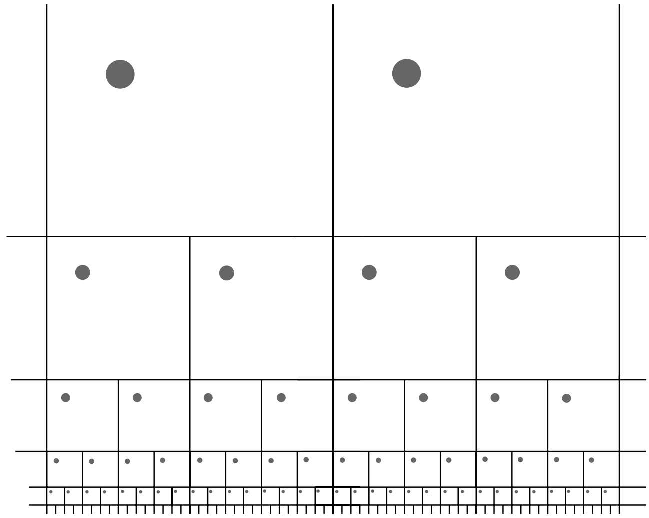


Figure 6. The binary tiling with a disk packing superimposed; primary tile vertices at i, $2+i$, $2i$, $2+2i$; disk centered at $\frac{1}{2}+\frac{7}{4}i$.

Unfortunately there are no $\mathcal{G}^2$-invariant probability measures on $\mathcal{P}_{\mathcal{B}}$ that assign these tilings positive measure. (This follows easily from the identification of a boundary for $\mathbb{H}^2$—which in the upper half-plane model is the union of the x-axis and a point at infinity—and the fact that PSL(2, $\mathbb{R}$) extends continuously to the boundary [**9**].) In fact, these tilings (but without the bumps) had figured in an earlier study of the density of certain disk packings by Böröczky [**4**]. Consider the disk packing indicated in Figure 6. In that packing each copy of a binary tile has one disk in it, suggesting a certain value for the "density" of the disk packing. However, if we apply the rigid motion $z \mapsto 6z/5$ to the disks, we get the result in Figure 7, in which every binary tile has two disks in it, suggesting a density twice the original value! This contradiction has had

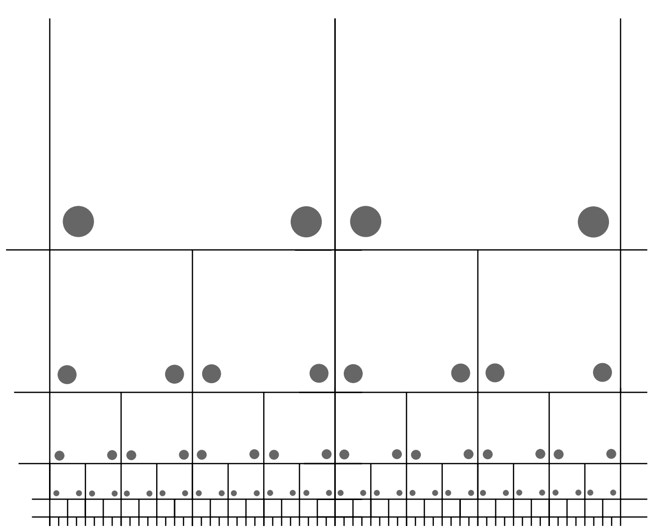


Figure 7. The binary tiling with a disk packing superimposed; primary tile vertices at i, $2+i$, $2i$, $2+2i$; disks centered at $\frac{3}{5}+\frac{21}{10}i$, $3+\frac{21}{10}i$.

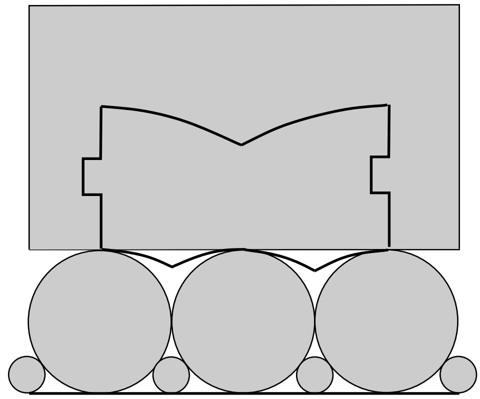


Figure 8. The modified binary tile in the upper half-plane.

a profound influence on those studying the density of packings in hyperbolic spaces (see [**15**]). It was to overcome this contradiction that the density formalism discussed in this article was introduced in [**8**] and [**9**].

As suggested, we can easily adapt the formalism used in the first part of this article to packings in hyperbolic spaces, the only nontrivial new step being the appropriate generalization of the Birkhoff theorem to apply to actions of PSL$(2, \mathbb{R})$ (or, more generally, to isometry groups of $\mathbb{H}^d$), which was only discovered by Nevo et al. in the last decade [**25**], [**26**].

In order to identify densest packings using this formalism, we need to find appropriate invariant measures. This can be done with modifications of Penrose's tile, as with the simple example shown in Figure 8. In this example certain of the bumps have been altered, and the basic shape stretched, for the following reason. Consider the packing of the plane by horoballs indicated in Figure 9, a packing that has symmetry

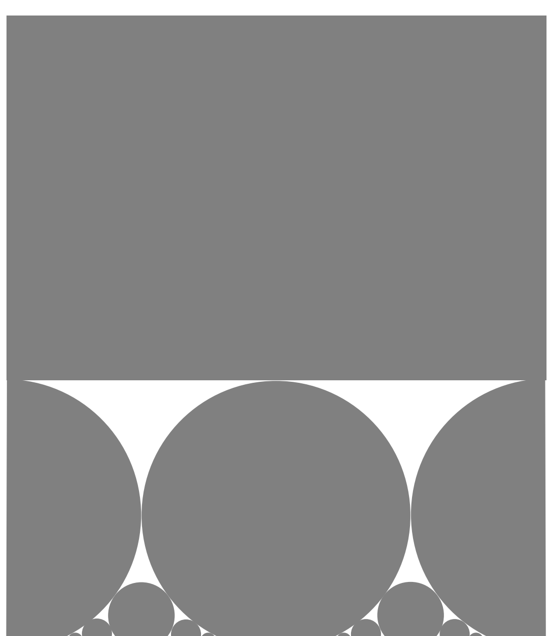


Figure 9. A packing of horoballs.

group PSL(2, $\mathbb{Z}$). As noted earlier, bodies like the Penrose binary tile "naturally" tile horoballs in a great diversity of ways. We can thus use them to tile each of the horoballs in Figure 9, the only complication being the bumps that stick out of the horoballs. This is the reason for the modification of the body, allowing these bumps to fill up precisely the regions between the horoballs.

Other modifications can be made, in at least the following two ways. First, we could start with a horoball packing in which n horoballs meet, leaving empty "n-gonal" regions and generalizing the case $n = 3$ just considered (see Figure 10). (The symmetry groups of such horoball packings are the Hecke groups $\Gamma[2\cos(\pi/n)]$, which are cofinite but not cocompact in PSL(2, $\mathbb{R}$) [**21**].) Also, instead of having only two outgoing bumps on the body, we can have any number, as suggested in Figure 11. These variations yield a two-parameter family of packing problems, for each of which the underlying horoball packing can be used to prove the existence of a $\mathcal{G}^2$-invariant probability measure that gives full measure to the tilings, thus proving that the generic tilings are actually "densest packings" in the sense of the formalism. (Those tilings analogous to the original Penrose tilings of Figure 3a are not generic points.)

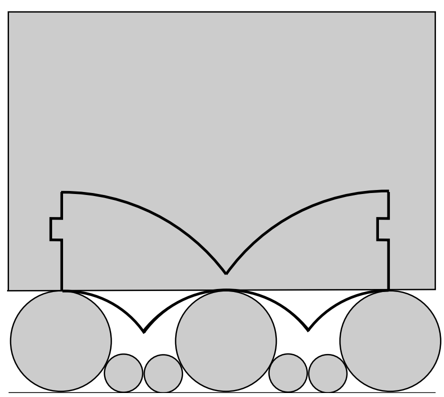


Figure 10. A "pentagonal" binary tile.

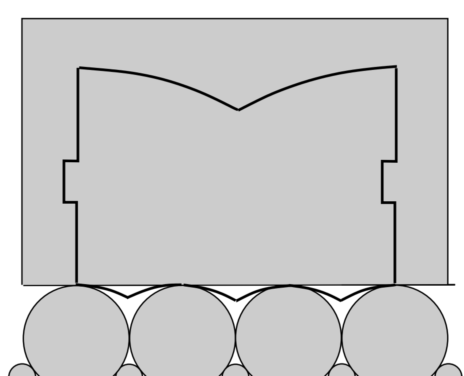


Figure 11. A triangular tile with three prongs.

As was the case for Euclidean packings with nonunique solution, in order to classify the foregoing tilings through their behavior under the action of $\mathcal{G}^2 = \mathrm{PSL}(2, \mathbb{R})$ we are led to investigate *non-cocompact* subgroups of $\mathcal{G}^2$, in this case the Hecke groups.

Results were obtained on general properties of actions of such groups through investigation of the cusp point set of the corresponding horoball packing [**7**].

Another result in this direction was recently obtained by Bowen [**6**]. It is a simple fact that a densest packing P in Euclidean space can always be well approximated by packings P_n with cocompact symmetry, where the density of P_n approaches that of P. This is not so obvious for densest packings in hyperbolic space, but has recently been shown to hold at least for packings by congruent disks in $\mathbb{H}^2$. This can be interpreted, in some sense, as proof of a kind of denseness of the cocompact subgroups among the subgroups of PSL(2, $\mathbb{R}$).

The modifications cited of Penrose's binary tile yield the only packing problems that have been solved in a hyperbolic space, aside from those with unique solution (and therefore cocompact symmetry), such as the packings of $\mathbb{H}^2$ by disks with special values of the radius (called "tight" in [**8**]). For these a proof like that of Fejes Tóth was given by Böröczky [**5**]. However, there is some symmetry information available about ball packings in $\mathbb{H}^d$ that is not known for $\mathbb{E}^d$. Specifically, if $\mathcal{B}$ consists of a single hyperbolic ball of radius R, then for all but countably many R there are *no* densest packings with cocompact symmetry [**8**]. This sort of behavior may well be common for packing problems in $\mathbb{E}^d$ but so far it has only been found in the Euclidean setting for polyhedra with complicated bumps on their faces (e.g., the kite & dart tilings), not for such natural bodies as balls.

4. CONCLUSION. We have tried to illustrate in this article how the discovery of aperiodic tilings by Berger in 1966 [**1**] and the unusual way their symmetries are manifested has led to new techniques and renewed interest in the study of the qualitative (symmetry) properties of densest packings. The symmetries of densest packings have been of interest since well before Hilbert put foward his eighteenth problem, but until recently it was expected that the symmetries of such packings would have compact fundamental domains, as in Figure 1. As we have seen, aperiodic packing problems exhibit their symmetries in more subtle ways, and lead to the much broader study of groups of rigid motions without compact fundamental domains.

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CHARLES RADIN started as an art student at the High School of Music and Art, the School of Visual Arts, and the Art Students League in New York City. He then obtained degrees in physics—a B.S. from the City College of New York in 1965 and a Ph.D. from the University of Rochester in 1970—leading to postdoctoral positions at the University of Nijmegen, Princeton University, Rockefeller University, and the University of Pennsylvania. He is currently professor of mathematics at the University of Texas at Austin, where he has been since 1976. His research interests have ranged from mathematical physics to operator algebras, ergodic theory, and discrete geometry.
University of Texas at Austin, TX 78712-1082
radin@math.utexas.edu