

SURFACE EFFECTS IN DENSE RANDOM GRAPHS WITH SHARP EDGE CONSTRAINT

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ABSTRACT. We show that the random number T_n of triangles in a random graph on n vertices, with a strict constraint on the total number of edges, admits an expansion $T_n = an^3 + bn^2 + F_n$, where a and b are numbers, with the mean $\langle F_n \rangle = O(n)$ and the standard deviation $\sigma(T_n) = \sigma(F_n) = O(n^{3/2})$. The presence of a ‘surface term’ bn^2 has a significance analogous to the macroscopic surface effects of materials, and is missing in the model where the edge constraint is removed. We also find the surface effect in other graph models using similar edge constraints.

1. A RANDOM GRAPH MODEL WITH DEPENDENT EDGES

Consider the spaces G_n , $n = 1, \dots$, of simple graphs on n labeled vertices, on which we will define probability distributions, giving us random graph models of increasing ‘size’ n . Let H be an arbitrary but fixed graph, and for $g \in G_n$ let $T_H(g)$ denote the number of copies of H found in g . We will compute the growth rates of the expectation and variance of T_H , and will show that the expectation has both ‘volume’ and ‘surface’ rates of growth, which are not overshadowed by the lower rate of growth of its standard deviation. This is analogous to the volume and surface components of macroscopic materials, and indeed our models were chosen to mimic the statistical mechanics model of macroscopic materials.

We define our probability distributions as follow. For each $0 < p < 1$ fix some sequence $E_n = p \binom{n}{2} + O(1)$ and define $\mu_n(p)$ as the uniform distribution on those $g \in G_n$ such that the total number of edges, denoted $T_e(g)$, is exactly equal to E_n . Graphs in this ensemble are often called Erdős-Rényi graphs [3]; see [1, 2] for a broad overview.

Let H be a fixed graph with v vertices and $\ell > 1$ edges. (For simplicity we assume every vertex in H lies on at least one edge.) Fixing the distribution $\mu_n(p)$, we are interested in the expectation $\langle T_H \rangle_{n,p}$ and the variance $Var(T_H)_{n,p}$. We first need some specialized notation to simplify the statements of the results.

Let $N_n = \binom{n}{2}$. For any positive integer k , let $P(E_n, N_n, k) = \frac{E_n(E_n - 1) \cdots (E_n + 1 - k)}{N_n(N_n - 1) \cdots (N_n + 1 - k)}$. Let c_n be the number of copies of H that appear in the complete graph; specifically, $c_n = n(n - 1) \cdots (n + 1 - v) / |S_H|$, where S_H is the group of symmetries of the graph H . For instance, if H is a triangle, then $c_n = n(n - 1)(n - 2)/6$, while if H is a “2-star” (that is, a graph with three vertices and two edges), then $c_n = n(n - 1)(n - 2)/2$.

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For each integer k between 0 and ℓ , let C_k be the number of times in which two distinguishable copies of H in the complete graph share exactly k edges. Note that

$$(1) \quad \sum_{k=0}^{\ell} C_k = c_n^2; \quad \sum_{k=0}^{\ell} k C_k = \frac{\ell^2 c_n^2}{N_n}.$$

The second equation comes from the fact that each of the ℓ edges in the first copy of H has probability $1/N_n$ of being the same as each of the ℓ edges of the second copy. The two sides are just different expressions for the sum, over all configurations, of the number of shared edges. Note also that C_0 is of order n^{2v} , C_1 is of order n^{2v-2} , C_2 and C_3 are of order n^{2v-3} , and all other terms are of order n^{2v-4} or smaller.

Theorem 1.1. *The expectation and variance of T_H are given by*

$$(2) \quad \begin{aligned} \langle T_H \rangle_{n,p} &= c_n P(E_n, N_n, \ell) \\ &= c_n p^\ell + O(n^{v-2}) = \frac{1}{|S_H|} [n^v p^\ell - \frac{v(v-1)}{2} n^{v-1} p^\ell] + O(n^{v-2}), \\ \text{Var}(T_H)_{n,p} &= C_2 p^{2v-2} (1-p)^2 + C_3 p^{2v-3} (1-3p^2+2p^3) + O(n^{2v-4}) \\ &= O(n^{2v-3}). \end{aligned}$$

If H does not contain any triangles, then the C_3 term in the formula for $\text{Var}(T_H)_{n,p}$ is itself $O(n^{2v-4})$ and can be ignored.

In particular, the standard deviation of T_H has a lower growth rate, $O(n^{v-3/2})$, than that of the $O(n^{v-1})$ second term in the expansion of T_H , implying a meaningful surface effect.

Proof. The formula for the expectation is easy. Each of the c_n configurations has probability $P(E_n, N, \ell)$ of appearing. We also note that

$$(3) \quad \begin{aligned} P(E_n, N_n, k) &= \frac{E_n(E_n-1)\cdots(E_n+1-k)}{N_n(N_n-1)\cdots(N_n+1-k)} \\ &= \left(\frac{E_n}{N_n}\right)^k \frac{(1-\frac{1}{E_n})\cdots(1-\frac{k-1}{E_n})}{(1-\frac{1}{N_n})\cdots(1-\frac{k-1}{N_n})} \\ &= \left(\frac{E_n}{N_n}\right)^k \frac{1-\frac{k(k-1)}{2E_n} + O(n^{-4})}{1-\frac{k(k-1)}{2N_n} + O(n^{-4})} \\ &= \left(\frac{E_n}{N_n}\right)^k \left(1 + \frac{k(k-1)}{2N_n} \left(1 - \frac{N_n}{E_n}\right)\right) + O(n^{-4}) \\ &= p_n^k - \frac{k(k-1)}{2N_n} (p_n^{k-1} - p_n^k) + O(n^{-4}), \end{aligned}$$

where $p_n \equiv E_n/\binom{n}{2}$. Since c_n is $O(n^v)$, this implies that $c_n P(E_n, N_n, \ell) = c_n p_n^\ell + O(n^{v-2})$.

Next we compute the variance. The expected value of T_H^2 is obtained by writing down all the configurations of two H 's, and adding their probabilities. That is,

$$(4) \quad \langle T_H^2 \rangle_{n,p} = \sum_{k=0}^{\ell} C_k P(E_n, N_n, 2\ell - k).$$

Meanwhile,

$$\begin{aligned}
\langle T_H \rangle_{n,p}^2 &= c_n^2 P(E_n, N_n, \ell)^2 \\
&= c_n^2 ((P(E_n, N_n, \ell)^2 - P(E_n, N_n, 2\ell)) + c_n^2 P(E_n, N_n, 2\ell)) \\
(5) \quad &= c_n^2 ((P(E_n, N_n, \ell)^2 - P(E_n, N_n, 2\ell)) + \sum_k C_k P(E_n, N_n, 2\ell)).
\end{aligned}$$

However, by equation (3),

$$(6) \quad P(E_n, N_n, \ell)^2 - P(E_n, N_n, 2\ell) = \frac{\ell^2}{N_n} (p_n^{2\ell-1} - p_n^{2\ell}) + O(n^{-4}),$$

but by (1), $\ell^2 c_n^2 / N_n = \sum_{k=0}^{\ell} k C_k$, so

$$(7) \quad \langle T_H \rangle_{n,p}^2 = \sum_{k=0}^{\ell} C_k [P(E_n, N_n, 2\ell) + k(p_n^{2\ell-1} - p_n^{2\ell}) + O(n^{-4})].$$

This makes the variance

$$(8) \quad \text{Var}(T_H)_{n,p} = \sum_{k=0}^{\ell} C_k [P(E_n, N_n, 2\ell - k) - P(E_n, N_n, 2\ell) - k(p_n^{2\ell-1} - p_n^{2\ell})] + O(n^{2v-4}).$$

The $k = 0$ term is identically zero. All the other C_k 's are at most $O(n^{2v-2})$, so we can use the approximations $P(E_n, N_n, 2\ell - k) = p_n^{2\ell-k} + O(n^{-2})$ and $P(E_n, N_n, 2\ell) = p_n^{2\ell} + O(n^{-2})$ to get

$$\begin{aligned}
\text{Var}(T_H)_{n,p} &= \sum_{k=1}^{\ell} C_k (p_n^{2\ell-k} - p_n^{2\ell} - k(p_n^{2\ell-1} - p_n^{2\ell})) + O(n^{2v-4}) \\
(9) \quad &= \sum_{k=1}^{\ell} C_k (p^{2\ell-k} - p^{2\ell} - k(p^{2\ell-1} - p^{2\ell})) + O(n^{2v-4}),
\end{aligned}$$

since $p_n = p + O(n^{-2})$. In this last sum the $k = 1$ term is zero, the $k = 2$ term is $C_2 p^{2\ell-2} (1 - p)^2$, the $k = 3$ term is $C_3 p^{2\ell-3} (1 - 3p^2 + 2p^3)$, and all remaining terms are $O(n^{2v-4})$ or smaller.

Finally, C_3 is only of order n^{2v-3} if H contains triangles. If H does not contain triangles, then the only way for two copies of H to have three edges in common is to have four or more vertices in common. Thus, if H does not contain triangles, then $C_3 = O(n^{2v-4})$ and we are left with

$$(10) \quad \text{Var}(T_H)_{n,p} = C_2 p^{2\ell-2} (1 - p)^2 + O(n^{2v-4}).$$

□

Simple use of Chebychev's inequality elucidates the terms of different growth:

Corollary 1.2.

$$(11) \quad \frac{T_H(n)}{n^v} \rightarrow \frac{1}{|S_H|} p^\ell, \quad n \left[\frac{T_H(n)}{n^v} - \frac{1}{|S_H|} p^\ell \right] \rightarrow \frac{1}{|S_H|} \frac{v(v-1)}{2} p^\ell$$

where the random variables are converging in probability.

We now address the choice we made to use n to measure the ‘size’ of our random system G_n , which was then used when identifying ‘surface’ effects. The probability distribution on G_n is based on fixing the number of edges that can appear in the graphs of G_n which we allow, the graphs which appear in our analysis. In this sense the size of G_n is perhaps more properly $N_n = \binom{n}{2}$, as the constraint strictly limits the fraction of the possible N_n possible edges. If we rewrite our expansions of the mean and variance of T_H in powers of N_n we get:

$$(12) \quad \begin{aligned} \langle T_H \rangle_{n,p} &= \frac{1}{|S_H|} [2^{\frac{v}{2}} N_n^{\frac{v}{2}} - 2^{\frac{v}{2}-\frac{3}{2}} v (v-2) N_n^{\frac{v}{2}-\frac{1}{2}}] + O(N_n^{\frac{v}{2}-1}), \\ \text{Var}(T_H)_{n,p} &= O(N_n^{v-\frac{3}{2}}). \end{aligned}$$

This decomposition of $\langle T_H \rangle_{n,p}$ is somewhat different from that of equation (2), but the standard deviation of T_H still has a growth rate, $N_n^{\frac{v}{2}-\frac{3}{4}}$, that is smaller than the subleading term in the expansion of the mean of T_H . The precise size of the surface term depends on the choice of size parameter, but the existence of a surface effect is unambiguous.

We will address this issue again in the next section, and again in the Conclusion.

2. A RANDOM GRAPH MODEL WITH INDEPENDENT EDGES

Now we turn to the model defined by having all edges appear independently with probability p . (This model is also often called ‘Erdős-Rényi’, despite being introduced in [4].) If one identifies edges with coin flips the model can be understood as a coin flip model in which one focuses on random variables T_H that are not easily described in the standard setting of coin flips. This presentation makes it easy to see how adding dependence to the coin flips, through fixing the fraction of heads, affects these ‘graph theoretic’ random variables.

In this model, the total number T_e of edges is a random variable with mean $N_n p$ and variance $N_n p(1-p)$. However, this model can also be used to mimic the model of the last section with a sharp constraint on the number of edges, using a residual variance, as we shall see. The variable T_H is correlated with T_e , with correlation coefficient

$$(13) \quad r = \frac{\text{Cov}(T_H, T_e)_{n,p}}{\sqrt{\text{Var}(T_H)_{n,p} \text{Var}(T_e)_{n,p}}}.$$

A common interpretation of r is that a fraction r^2 of the variance of T_H in the dependent-edge model can be ‘explained’ by the correlation with T_e , and that the remaining *residual variance* of T_H is

$$(14) \quad \text{ResVar}(T_H)_{n,p} = (1 - r^2) \text{Var}(T_H)_{n,p} = \text{Var}(T_H)_{n,p} - \frac{\text{Cov}(T_H, T_e)_{n,p}^2}{\text{Var}(T_e)_{n,p}}.$$

If we model T_H as a linear function of T_e plus a residual piece that is uncorrelated to T_e , then $\text{ResVar}(T_H)_{n,p}$ is the variance of this residual piece. That is, $\text{ResVar}(T_H)_{n,p}$ is the variance we should expect if we further constrain our system to have a specific value of T_e , as in the previous section.

Theorem 2.1. *In the independent-edge model, the expectation, variance, and residual variance of T_H are given by:*

$$\begin{aligned}
\langle T_H \rangle_{n,p} &= c_n p^\ell = \frac{1}{|S_H|} \left[n^v - \frac{v(v-1)}{2} n^{v-1} \right] p^\ell + O(n^{v-2}) \\
\text{Var}(T_H)_{n,p} &= \sum_k C_k (p^{2\ell-k} - p^{2\ell}) = O(n^{2v-2}) \\
\text{ResVar}(T_H)_{n,p} &= \sum_k C_k (p^{2\ell-k} - p^{2\ell} - k(p^{2\ell-1} - p^{2\ell})) \\
&= C_2 p^{2\ell-2} (1-p)^2 + C_3 p^{2\ell-3} (1-3p^2+2p^3) + O(n^{2v-4}) \\
(15) \qquad \qquad &= O(n^{2v-3}).
\end{aligned}$$

The independent-edge model gives the same results for the mean of T_H , up to unimportant lower-order corrections, as the dependent-edge model. However the variance is one power of n larger than in the dependent-edge model, so the subleading term in the expansion of the mean of T_H is of the same order, $O(n^{v-1})$, as the standard deviation of T_H , and we say the independent-edge model does not have a surface term. Not surprisingly, the *residual* variance of T_H in the independent-edge model matches the variance of T_H in the dependent-edge model.

Proof. The calculation is essentially the same as in the dependent-edge model, only with $P(E_n, N_n, k)$ replaced by p^k . Since there are c_n configurations for H , each with probability p^ℓ , the expectation of T_H is $c_n p^\ell$. We then have

$$(16) \qquad \qquad \langle T_H \rangle_{n,p}^2 = c_n^2 p^{2\ell} = \sum_{k=0}^{\ell} C_k p^{2\ell}.$$

As for $\langle T_H^2 \rangle_{n,p}$, each of the configurations with k overlapping edges has probability $p^{2\ell-k}$, so

$$(17) \qquad \qquad \langle T_H^2 \rangle_{n,p} = \sum_{k=0}^{\ell} C_k p^{2\ell-k}.$$

Subtracting, we get

$$(18) \qquad \qquad \text{Var}(T_H)_{n,p} = \langle T_H^2 \rangle_{n,p} - \langle T_H \rangle_{n,p}^2 = \sum_{k=1}^{\ell} C_k (p^{2\ell-k} - p^{2\ell}).$$

This sum is dominated by the $k=1$ term, which scales as n^{2v-2} .

To get the covariance of T_H and T_e we must compute the number of ways to have an H and a special edge (representing T_e). There are $c_n(N_n - \ell)$ ways to have the edge be disjoint from the edges of H , and $c_n \ell$ ways to have the special edge be one of the edges of H . Thus

$$\begin{aligned}
\langle T_H T_e \rangle_{n,p} &= c_n (N_n - \ell) p^{\ell+1} + c_n \ell p^\ell, \\
\langle T_H \rangle_{n,p} \langle T_e \rangle_{n,p} &= c_n (N_n - \ell) p^{\ell+1} + c_n \ell p^{\ell+1}, \\
(19) \qquad \text{Cov}(T_H, T_e)_{n,p} &= c_n \ell (p^\ell - p^{\ell+1}) = c_n \ell p^\ell (1-p).
\end{aligned}$$

We then have

$$(20) \qquad \qquad \frac{\text{Cov}(T_H, T_e)_{n,p}^2}{\text{Var}(T_e)_{n,p}} = \frac{c_n^2 \ell^2}{N_n} p^{2\ell-1} (1-p).$$

However, $c_n^2 \ell^2 / N = \sum_{k=0}^{\ell} k C_k$, so

$$(21) \quad \text{Var}(T_H)_{n,p} - \frac{(\text{Cov}(T_H, T_e)_{n,p})^2}{\text{Var}(T_e)_{n,p}} = \sum_{k=0}^{\ell} C_k (p^{2\ell-k} - p^{2\ell} - k(p^{2\ell-1} - p^{2\ell})).$$

The $k = 0$ and $k = 1$ terms are identically zero, the terms with $k > 3$ are of order $O(n^{2v-4})$ or smaller, and what is left is $C_2 p^{2\ell-2} (1-p)^2 + C_3 p^{2\ell-3} (1-3p^2+2p^3)$. \square

Note that some of the equations in the theorem do not need lower-order corrections. The expectation agrees with the dependent-edge model up to order $O(n^{v-2})$, while the *residual* variance of the independent-edge model agrees with the variance of the dependent-edge model up to order $O(n^{2v-4})$.

The scale and relative lack of statistical significance of the surface term is unaffected by the choice of measure of the size of G_n . In terms of N_n , we have

$$(22) \quad \begin{aligned} \langle T_H \rangle_{n,p} &= \frac{1}{|S_H|} [2^{\frac{v}{2}} N_n^{\frac{v}{2}} - 2^{\frac{v}{2}-\frac{3}{2}} v (v-2) N_n^{\frac{v}{2}-\frac{1}{2}}] + O(N_n^{\frac{v}{2}-1}), \\ \text{Var}(T_H)_{n,p} &= O(N_n^{v-1}), \\ \text{ResVar}(T_H)_{n,p} &= O(N_n^{v-\frac{3}{2}}), \end{aligned}$$

so the standard deviation of T_H has a growth rate, $N_n^{\frac{v}{2}-\frac{1}{2}}$, equal to that of the second term in the expansion of the mean of T_H .

3. 2-STARS, TRIANGLES AND SQUARES

Now we work out three examples, specifically where H is a graph with 3 vertices and 2 edges (often called a “2-star” or a “cherry”), where H is a triangle, and where H is a square.

3.1. 2-stars. If H is a 2-star, then $c_n = n(n-1)(n-2)/2 = (n^3 - 3n^2 + 2n)/2$, $C_2 = c_n$, and $C_3 = 0$. Thus the expectation and variance in the dependent-edge model (i.e. the first model) are

$$(23) \quad \begin{aligned} \langle T_H \rangle_{n,p} &= c_n P(E_n, N_n, 2) \\ &= \left(\frac{n^3}{2} - \frac{3n^2}{2} + O(n) \right) (p^2 + O(n^{-2})) \\ &= \frac{p^2}{2} n^3 - \frac{3p^2}{2} n^2 + O(n), \\ \text{Var}(T_H)_{n,p} &= C_2 p^2 (1-p)^2 + O(n^2) \\ &= \frac{p^2 (1-p)^2}{2} n^3 + O(n^2). \end{aligned}$$

For the independent-edge model, we also need to compute C_1 , which works out to equal $2n(n-1)(n-2)(n-3) = 2n^4 - 12n^3 + O(n^2)$. The variance is then

$$\begin{aligned} \text{Var}(T_H)_{n,p} &= C_1 (p^5 - p^6) + C_2 (p^4 - p^6) \\ &= (2n^4 - 12n^3) (p^5 - p^6) + \frac{n^3}{2} (p^4 - p^6) + O(n^2) \end{aligned}$$

$$(24) \quad = 2(p^5 - p^4)n^4 + \frac{p^4 - 24p^5 + 23p^6}{2}n^3 + O(n^2),$$

and the residual variance is

$$(25) \quad \begin{aligned} \text{ResVar}(T_H)_{n,p} &= C_2 p^2 (1-p)^2 \\ &= \frac{p^2 (1-p)^2}{2} n^3 + O(n^2). \end{aligned}$$

3.2. Triangles. When H is a triangle, our relevant combinatorial factors are:

$$(26) \quad \begin{aligned} c_n &= \frac{n(n-1)(n-2)}{6} = \frac{n^3}{6} - \frac{n^2}{2} + O(n), \\ C_1 &= \frac{n(n-1)(n-2)(n-3)}{2} = \frac{n^4}{2} - 3n^3 + O(n^2), \\ C_2 &= 0, \\ C_3 &= c_n = \frac{n^3}{6} + O(n^2). \end{aligned}$$

In the dependent-edge model, we have

$$(27) \quad \begin{aligned} \langle T_H \rangle_{n,p} &= c_n P(E_n, N_n, 3) \\ &= \frac{p^3}{6} n^3 - \frac{p^3}{2} n^2 + O(n), \\ \text{Var}(T_H)_{n,p} &= C_3 p^3 (1 - 3p^2 + 2p^3) + O(n^2) \\ &= \frac{p^3 (1 - 3p^2 + 2p^3)}{6} n^3 + O(n^2). \end{aligned}$$

In the independent-edge model we have

$$(28) \quad \begin{aligned} \langle T_H \rangle_{n,p} &= c_n p^3 \\ &= \frac{p^3}{6} n^3 - \frac{p^3}{2} n^2 + O(n), \\ \text{Var}(T_H)_{n,p} &= C_1 (p^5 - p^6) + C_3 (p^3 - p^6) \\ &= \frac{p^5 - p^6}{2} n^4 + \frac{p^3 - 18p^5 + 17p^6}{6} n^3 + O(n^2), \\ \text{ResVar}(T_H)_{n,p} &= C_3 p^3 (1-p)^2 (1+2p) \\ &= \frac{p^3 (1-p)^2 (1+2p)}{6} n^3 + O(n^2). \end{aligned}$$

3.3. Squares. If H is a square, then $c_n = n(n-1)(n-2)(n-3)/8$, since we are picking 4 points and the group of symmetries of the square is the dihedral group of order 8. We then compute

$$\begin{aligned} C_1 &= \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2} \\ &= \frac{n^6}{2} - \frac{15n^5}{2} + O(n^3), \\ C_2 &= \frac{n(n-1)(n-2)(n-3)(n-4)}{2} + \frac{n(n-1)(n-2)(n-3)}{4} \\ &= \frac{n^5}{2} + O(n^4), \\ C_3 &= 0, \end{aligned}$$

$$(29) \quad C_4 = c_n = O(n^4).$$

The first term in C_2 comes from having two consecutive edges shared across the two squares, while the second comes from sharing non-consecutive edges.

In the dependent-edge model we then have

$$(30) \quad \begin{aligned} \langle T_H \rangle_{n,p} &= c_n P(E_n, N_n, 4) \\ &= \frac{p^4}{8} n^4 - \frac{3p^4}{4} n^3 + O(n^2), \\ \text{Var}(T_H)_{n,p} &= C_2 p^6 (1-p)^2 + O(n^4) \\ &= \frac{p^6 (1-p)^2}{2} n^5 + O(n^4). \end{aligned}$$

In the independent-edge model, we have

$$(31) \quad \begin{aligned} \langle T_H \rangle_{n,p} &= c_n p^4 \\ &= \frac{p^4}{8} n^4 - \frac{3p^4}{4} n^3 + O(n^2), \\ \text{Var}(T_H)_{n,p} &= C_1 (p^7 - p^8) + C_2 (p^6 - p^8) + C_4 (p^4 - p^8) \\ &= \frac{p^7 - p^8}{2} n^6 + \frac{p^6 - 15p^7 + 14p^8}{2} n^5 + O(n^4), \\ \text{ResVar}(T_H)_{n,p} &= C_2 p^6 (1-p)^2 + O(n^4) \\ &= \frac{p^6 (1-p)^2}{2} n^5 + O(n^4). \end{aligned}$$

4. BLOCK MODELS

In this section we sketch a more complex version of the previous models, in which there are vertices of various colors. More specifically, we consider colored graphs on B colors, where the number n_1, \dots, n_B of vertices of each color is fixed. We imagine a limit in which all the n_i 's go to infinity along a fixed line in \mathbb{R}^B . In the dependent-edge version of this model, we fix the number $E_{n,ij}$ of edges between vertices of colors i and j . In the independent-edge version of this model, we fix the probability p_{ij} of each such edge.

In the interest of brevity, we merely sketch the results. (Precise statements and proofs will appear in a subsequent paper.) We have

$$(32) \quad T_H = \sum_{\alpha} T_{H_{\alpha}},$$

where α indexes all the possible colorings of H . Each $T_{H_{\alpha}}$ has its expectation and variance described by expansions similar to (2) or (15), and similar formulas apply to the covariances of different H_{α} 's. As before, $\langle T_{H_{\alpha}} \rangle_{n,p}$ always scales as n^v , while the (co)variances of the $T_{H_{\alpha}}$'s in the dependent-edge model scale as n^{2v-3} , as do the residual (co)variances in the independent-edge model. The total (co)variances in the independent-edge model scale as n^{2v-2} . As before, the expectations are the same in the two models (up to $O(n^{v-2})$ corrections), and the residual variance in the independent-edge model equals the variance in the dependent-edge model, up to $O(n^{2v-4})$ corrections.

The combinatorial factors c_n, C_0, C_1 , etc. are different for different values of α , as are the probabilistic functions that replace p^{ℓ} and $p^{\ell-k}$. As a result, $\langle T_H \rangle_{n,p}$ cannot be written

as a single function of the n_i 's times a single function of the p_{ij} 's. To get an asymptotic understanding of $\langle T_H \rangle_{n,p}$, it is necessary to isolate all the different terms that are bigger than the standard deviation. That is, the leading terms of order n^v and the surface corrections of order n^{v-1} .

In the dependent-edge model, the subleading terms in the expansion of $\langle T_H \rangle_{n,p}$ are $O(\sqrt{n})$ larger than the standard deviation. Regardless of whether we measure the size of our system in terms of n , $\binom{n}{2}$, or some other yardstick, there is an unambiguous surface effect. By contrast, in the independent-edge model the subleading terms in the expansion of $\langle T_H \rangle_{n,p}$ are of the same order as the standard deviation.

5. CONCLUSION

We considered a sequence G_n of spaces of random graphs through which we study the growth rates of certain random counts, for instance triangles. The probability distributions on G_n are defined by strongly restricting the count of edges, and this restriction turns out to reduce the randomness in the counts of triangles, and indeed any other graph H , to such an extent that a surface phenomenon is produced (Theorem (1.1)): a lower order constant correction to the mean of the count of H , with growth rate larger than that of the fluctuations. Without the constraint there is no surface effect (Theorem (2.1)).

This work was motivated by previous studies of random graph models in which the randomness is produced by restrictions on the counts of two or more graphs, say both edges and triangles, and then counts of other graphs H are studied [7, 8, 9, 5, 10, 6, 11]. (When one has two or more count restrictions they can interfere and produce ‘phase transitions’, drastic sensitivity in the *highest order terms* of counts for H , encoded in what is called the entropy.) In those random graph models the highest order terms in the counts of graphs H turn out to be easily computable because the highest order terms are represented by block models [6]. This is one of the reasons we have included block models in Section 4.

Some of that modelling, for instance the edge/triangle model, was explicitly performed to help understand features in statistical mechanics. Statistical mechanics was created by Boltzmann and Gibbs based on two conservation laws, the fact that the sum of the energies of all the particles, and the sum of the masses of all the particles, are dynamically conserved and therefore can each be rigorously fixed as adjustable parameters. The way we produced the probability distribution on our G_n is an explicit copy of this, but only using the mass conservation. We would have liked to restrict two or more graphs (to study phase transitions) but were not able to control the combinatorics to look for surface effects when the leading order terms were so sensitive.

What was done here could all be done, in principle, in other combinatorial settings, for instance the sequence of spaces P_n of permutations on n objects. There is some literature [12] on random permutations in which constraints are put on the counts of two or more ‘patterns’, in order to study interactions between the constraints in the highest order terms in the expansions of the counts of other patterns, i.e. phase transitions. It would be of interest to explore the existence of surface effects in random pattern counts using only one pattern restriction.

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