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Notes

A Noncommutative L_1-Mean Ergodic Theorem

We show that operator algebras, as opposed to Banach lattices, provide the more natural structure for L_1-mean ergodic theory.

We will note below how a result in operator algebras yields a significant generalization of the following classical *L_1-mean ergodic theorem* of Kakutani.

THEOREM [1]. *Let (X, m) be a finite measure space, let $L = L_1(X, m)$ (the complex Banach space), and let T be a positive linear contraction on L such that $T\mathbb{1} = \mathbb{1}$. Then for every f in L there exists $\bar{f}$ in L such that in the norm topology*

$$1/N \sum_{n=0}^{N-1} T^n f \xrightarrow[N\to\infty]{} \bar{f}.$$

First we need some notation. If L is the predual of a W^*-algebra, let L_h (resp. L_+) be the real (resp. positive) part of L. Then we call an element u of L_+ a *unit* for L if

$$\bigcup_{n=1}^{\infty} \{g \in L_h \mid -nu \leqslant g \leqslant nu\}$$

is norm-dense in L_h. (A unit for L is just a faithful element u of L_+, faithful in the sense that for f in L^*, $u(f^*f) = 0$ implies $f = 0$.) Note that $\mathbb{1}$ is a unit for $L_1(X, m)$.

Considering $L_1(X, m)$ as the predual of the commutative W^*-algebra $L_\infty(X, m)$, the following is a noncommutative generalization of Kakutani's result.

THEOREM. *Let L be the predual of a W^*-algebra, and let u be a unit for L. Let T be a positive linear contraction on L such that $Tu = u$.*

Then for every f in L, there exists $\bar{f}$ in L such that in the norm topology

$$1/N \sum_{n=0}^{N-1} T^n f \xrightarrow[N\to\infty]{} \bar{f}.$$

Proof. From [2, Theorem II.2 (2)], the convex set $S_n(u)$ defined by

$$S_n(u) = \{g \in L_h \mid -nu \leqslant g \leqslant nu\}$$

is $\sigma(L, L^*)$-compact. Since $T\colon S_n(u) \to S_n(u)$, the standard argument due to Yosida (see the proof in [3]) yields the desired result for f in each $S_n(u)$, and thus of course for f in $\bigcup_n S_n(u)$. A simple ϵ-argument then gives the result for f in L_h, and then linearity gives the full result, for f in L. Q.E.D.

Note. There are numerous well-known ways to relax the hypotheses of Kakutani's theorem as stated above, but we consider the stated form to contain the essence of the result. Most, if not all, of these relaxations would trivially go through for our theorem.

The above is an extension of [4, Proposition 4], which is essentially the noncommutative generalization of Kakutani's theorem, but assuming (X, m) to be a *topological* measure space. It is shown in [4, 5] why these generalizations are natural and useful developments for quantum theory.

We emphasize that our generalization is a result of relaxing the usual order-theoretic assumptions (L_h is a lattice if and only if L^* is abelian, as one sees easily from [6]) in favor of algebraic structure.

References

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