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Average Boundary Conditions in Cauchy Problems*

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We consider the general Cauchy problem with initial data in a Hilbert space and with a formal dissipative linear generator. A complete parametrization is known of the (abstract) boundary conditions which make this problem well set. We exhibit a distinguished subset $\mathscr{B}_E$ of the set $\mathscr{B}$ of boundary conditions and demonstrate explicitly that the evolution associated with each B in $\mathscr{B}$ can be represented as a (time independent) average over the evolutions associated with B' in $\mathscr{B}_E$. Applications are discussed to Schrödinger equations in bounded regions or with singular potentials.

1. INTRODUCTION

Consider the Cauchy problem

$$(d/dt) f_t = Lf_t, \qquad t \geqslant 0 \tag{1}$$

where the initial data f_0 varies throughout a complex Hilbert space $\mathscr{H}$ (with inner product denoted $\langle \cdot, \cdot \rangle$) and where the linear operator L, with domain $D(L) \subseteq \mathscr{H}$, is dissipative, i.e.,

$$\langle Lf, f \rangle + \langle f, Lf \rangle \leqslant 0 \qquad \text{for all } f \text{ in } D(L).$$

In terms of the evolution operators U^t, $t \geqslant 0$, defined by the equation $U^t f_0 = f_t$, "solving" (1) means constructing the one parameter contraction semigroup (of evolution operators) with infinitesimal generator L, i.e., constructing the family $\{U^t \mid t \geqslant 0\}$ of operators on $\mathscr{H}$ with the properties

(i) $U^t U^s = U^{t+s}$,
(ii) $\| U^t \| \leqslant 1$, $t \geqslant 0$,
(iii) $\| U^t f - f \| \to 0$ as $t \to 0^+$,
(iv) $\|(U^t f - f)/t - Lf \| \to 0$ as $t \to 0$, for all f in $D(L)$,

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where $\|\cdot\|$ denotes the norm in $\mathscr{H}$ and also the operator norm for bounded operators on $\mathscr{H}$.

Now the above problem is well set only if there is a *unique* solution f_t of (1) for each f_0, and $t > 0$. However if L were, for example, a differential operator on a bounded region in $\mathbf{R}^n$, we would expect to have to choose from a variety of "auxiliary conditions" (traditionally *boundary* conditions) in order to uniquely specify the evolution.

Treating the abstract problem (1), Phillips has given in [1] a complete classification of all possible (abstract) auxiliary conditions (and therefore of their associated infinitesimal generators and evolution semigroups) for each of two situations of very general interest; where one requires the actual infinitesimal generator L to be either (a) an *extension* of a given "minimal operator" $L_{\min}$, or (b) a *restriction* of a given "maximal operator" $L_{\max}$. We will parametrize such extensions and restrictions by $B \in \mathscr{B}$.

As an example consider $\mathscr{H} = L_2(0, 1)$ and $L_{\min} = d/dx$ with $D(L_{\min})$ the infinitely differentiable functions with compact support in the open interval (0, 1). It is an easy exercise to show that the set of all contraction semigroups whose generators extend $L_{\min}$ can be parametrized by $\mathscr{B} = \{z \in \mathbf{C} \mid |z| \leqslant 1\}$ where

$$(U_z{}^t f)(x) = z^{[x+t]_I} f([x + t]_F) \tag{2}$$

and $[y]_I$ (resp. $[y]_F$) denotes the integral part (resp. fractional part) of y. Note that $U_z{}^t$ is invertible (i.e., unitary) if and only if $|z| = 1$.

In this paper, for each of the cases (a) and (b) we will topologize the parameter space $\mathscr{B}$, exhibit a *distinguished subset* $\mathscr{B}_E$ of $\mathscr{B}$ and then construct, for each B in $\mathscr{B}$, a regular Borel probability measure μ_B on $\mathscr{B}$, *concentrated on* $\mathscr{B}_E$, with the property that

$$U_B{}^t = \int U_{B'}^t \, d\mu_B(B') \qquad \text{for all} \quad t \geqslant 0.$$

In terms of (2) for example, $\mathscr{B}_E = \{z \in \mathscr{B} \mid |z| = 1\}$, and if $z = r\exp(i\theta)$, $0 \leqslant r \leqslant 1$, and $z' = \exp(i\theta')$, then

$$\begin{aligned} U_z{}^t = \int U_{z'}^t \, d\mu_z(z') &= \frac{1}{2\pi} \int_0^{2\pi} U_{z'(\varphi)}^t \, d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} U_{\exp[i\varphi(\theta')]}^t \left[\frac{1 - r^2}{|\exp(i\theta') + r|^2}\right] d\theta' \end{aligned}$$

where $\exp[i\varphi(\theta')] = \exp[i(\theta - \theta')][\exp(i\theta') + r]/[\exp(-i\theta') + r]$.

The fact that in this example the general noninvertible (i.e., time irreversible) evolution is represented as an average over invertible (i.e., time reversible) evolutions will be seen to be a consequence of the fact that $KL_zK = L_z^*$ (where * denotes operator adjoint) for each of the infinitesimal generators L_z, where K is a conjugation operator on $\mathscr{H}$ (in this example $(Kf)(x) = \overline{f(1 - x)}$.) Of course the average of distinguished semigroups with respect to a probability measure is not in general a semigroup. In this example, only the special measures μ_z give rise to semigroups.

Finally, connection is made with the work of Nelson [2] on Schrödinger equations with singular potentials.

2. Notation

Throughout this paper $\mathscr{H}$ will denote a fixed separable complex Hilbert space of dimension $\geqslant 1$ and I will denote the identity operator on $\mathscr{H}$. Given complex Hilbert spaces X and Y (of dimension $\geqslant 1$), $\mathscr{B}(X, Y)$ will denote the Banach space of continuous linear maps from X into Y, with open (resp. closed) ball $\mathscr{B}_r(X, Y)$ (resp. $\bar{\mathscr{B}}_r(X, Y)$) of radius r. Unless otherwise indicated $\mathscr{B}(X, Y)$ and its subsets will be assumed to be in the strong operator topology. If X_1 is a proper nonzero subspace of X and Y_1 is a nonzero subspace of Y, an element of $\mathscr{B}(X_1, Y_1)$ will often be identified, without comment or notational distinction, with its unique extension in $\mathscr{B}(X, Y)$ whose kernel contains $X_1^\perp = X \ominus X_1$ (the orthogonal complement in X of X_1). $\mathscr{B}(X, X)$ will be written $\mathscr{B}(X)$, P_X denotes the orthogonal projection on X_1, and $\mathscr{B}_E(X, Y)$ is defined to be the set $\{V \in \mathscr{B}(X, Y) \mid V^*V = P_X$ or $VV^* = P_Y$ or both$\}$, which is always nonempty. Note that $\mathscr{B}_E(X, Y)$ is the set of extreme points of the convex set $\bar{\mathscr{B}}_1(X, Y)$.

3. Calculation of a Choquet Boundary

Given B in $\bar{\mathscr{B}}_1(X, Y)$, where X and Y are nonzero subspaces of $\mathscr{H}$, consider the polar decomposition $B = UR$, where U in $\mathscr{B}(X, Y)$ is a partial isometry and $R = (B^*B)^{1/2}$ in $\mathscr{B}(X)$ satisfies $0 \leqslant R \leqslant P_X$. For z in $\mathbf{C}$, $|z| \leqslant 1$, we define $R(z)$ in $\mathscr{B}(X)$, by the functional calculus in R, as $f_z(R)$ where f_z is the following function on $[0, 1]$:

$$f_z(\lambda) = \frac{z+\lambda}{1+z\lambda}, \qquad z \neq -1, \qquad 0 \leqslant \lambda \leqslant 1,$$
$$= -1, \qquad z = -1, \qquad 0 \leqslant \lambda < 1,$$
$$= 1, \qquad z = -1, \qquad \lambda = 1.$$

Note that $R(z)$ is norm analytic in z for $|z| < 1$ and (strong operator) continuous in z for $|z| \leqslant 1$. (The former is a standard fact in operator theory [3, 4.3.1] and the latter follows easily using Lebesgue's dominated convergence theorem.) Furthermore, if $|z| = 1$, it is clear that $R(z)$ is unitary in $\mathscr{B}(X)$. Now if $U \notin \mathscr{B}_E(X, Y)$, i.e., $U^*U = P_F$ and $UU^* = P_G$ with $F \neq X$ and $G \neq Y$, choose any V in $\mathscr{B}_E(F^\perp, G^\perp)$ and define, for $|z| \leqslant 1$, $B(z) = UR(z) + zV$ and $B_n(z) = (1 - (1/n))\, B(z)$, $n = 1, 2,\ldots$. If $U \in \mathscr{B}_E(X, Y)$, define, for $|z| \leqslant 1$, $B(z) = B$ and $B_n(z) = (1 - (1/n))B$, $n = 1, 2,\ldots$. Then for any B in $\bar{\mathscr{B}}_1(X, Y)$, $B(0) = B$, while if $|z| = 1$, $B(z) \in \mathscr{B}_E(X, Y)$. Let $F(X, Y)$ be the algebra of those continuous functions f from $\bar{\mathscr{B}}_1(X, Y)$ into $\mathscr{B}(\mathscr{H})$ which are Gâteaux differentiable on $\mathscr{B}_1(X, Y)$ (i.e., $[f(A + zB)h - f(A)h]/z$ is Cauchy as $z \to 0$, for A, $B \in \mathscr{B}_1(X, Y)$ and $h \in \mathscr{H}$), and bounded in the sense that the range of f is contained in $\mathscr{B}_r(\mathscr{H})$ for some radius r (which may depend on f). We note using [3, 3.17.1] that such f are automatically Fréchet differentiable on $\mathscr{B}_1(X, Y)$, i.e., $[f(A + B)h - f(A)h]/\| B \|$ is Cauchy as $\| B \| \to 0$, for A, $B \in \mathscr{B}_1(X, Y)$, and $h \in \mathscr{H}$. If $f \in F(X, Y)$ and $B \in \bar{\mathscr{B}}_1(X, Y)$, then $f[B(z)] = \lim_{n\to\infty} f[B_n(z)]$ is continuous in z for $|z| \leqslant 1$ and, by Vitali's theorem [3, 3.14.1], differentiable in z for $|z| < 1$. Therefore by Cauchy's integral formula [3, 3.11.3], if Γ is the unit circle in $\mathbf{C}$,

$$f(B) = f[B(0)] = \frac{1}{2\pi i} \int_\Gamma \frac{f[B(z)]}{z}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f[B(\exp(i\theta))]\, d\theta. \quad (3)$$

Define $\mathscr{B}_B = \{B[\exp(i\theta)] \mid 0 \leqslant \theta \leqslant 2\pi\}$. Formula (3) is our main technical tool; we summarize the above argument in

LEMMA 1. *For each B in $\bar{\mathscr{B}}_1(X, Y)$, Eq. (3) explicitly defines a regular Borel probability measure, μ_B, on $\bar{\mathscr{B}}_1(X, Y)$,* concentrated on *$\mathscr{B}_B \subseteq \mathscr{B}_E(X, Y)$ and such that for all f in $F(X, Y)$*

$$f(B) = \int f(B')\, d\mu_B(B').$$

Remark. Each $\mathscr{B}_B$ is compact and therefore Borel measurable, and μ_B is concentrated on $\mathscr{B}_B$ in the sense that (by definition) $\mu_B(S) = \mu_B(S \cap \mathscr{B}_B)$ for every Borel measurable S.

Remark. Since $\mathscr{B}_E(X, Y)$ is the set of convex extreme points of $\bar{\mathscr{B}}_1(X, Y)$, and since the identity function is in $F(X, Y)$, $\mathscr{B}_E(X, Y)$ is the smallest subset of $\bar{\mathscr{B}}_1(X, Y)$ which can play the desired role in Lemma 1; thus $\mathscr{B}_E(X, Y)$ is the Choquet boundary of $F(X, Y)$.

4. Phillips' Classification

With any operator L on $\mathscr{H}$ which is dissipative, we can associate the Cayley transform $J = (I + L)(I - L)^{-1}$ with domain $D(J) = \mathrm{Ran}(I - L)$. Conversely, with any contraction J on $\mathscr{H}$, with domain $D(J)$, such that $\mathrm{Ran}(I + J)$ is dense in $\mathscr{H}$, we can associate the operator $L = (J - I)(J + I)^{-1}$ with $D(L) = \mathrm{Ran}(I + J)$. These facts are discussed by Phillips in [1] where he proves the two theorems:

THEOREM P_1. *If L is a dissipative operator with dense domain in $\mathscr{H}$, then its Cayley transform J is a contraction operator and* $\mathrm{Ran}(I + J)$ *is dense; L and J are closed together. Conversely, if J is a contraction operator with* $\mathrm{Ran}(I + J)$ *dense in $\mathscr{H}$, the associated operator L is a dissipative operator with dense domain and J is its Cayley transform. The pairing $J \leftrightarrow L$ is a one-to-one inclusion preserving correspondence between all dissipative extensions $\tilde{L}$ of L and all contraction extensions $\tilde{J}$ of J. In particular the maximal dissipative extensions $\tilde{L}$ of L correspond to the contraction extensions $\tilde{J}$ of J with $D(\tilde{J}) = \mathscr{H}$.*

THEOREM P_2. *An operator $\tilde{L}$ is the infinitesimal generator of a contraction semigroup on $\mathscr{H}$ if and only if $\tilde{L}$ is a maximal dissipative operator with dense domain.*

Before we continue it will be convenient to reparametrize the above classification by means of a simple geometric lemma, which we will prove by a variation of the proof of the polar decomposition. From here on L will always denote a fixed closed dissipative operator with dense domain $D(L) \subseteq \mathscr{H}$, J will be its Cayley transform with closed domain denoted M, and we will be describing features of those extensions $\tilde{J}$ of J such that the associated $\tilde{L}$ generates a contraction semigroup, i.e., the contraction extensions $\tilde{J}$ of J with $D(\tilde{J}) = \mathscr{H}$.

Since J is in $\bar{\mathscr{B}}_1(M, \mathscr{H})$, $\tilde{J}$ in $\bar{\mathscr{B}}_1(\mathscr{H})$ extends J if and only if $\tilde{J} = J + V$ where $V \in \bar{\mathscr{B}}_1(M^\perp, \mathscr{H})$ and $\| \tilde{J}^* \| \leqslant 1$. The last condition is equivalent to

$$\| J^*f \|^2 + \| V^*f \|^2 \leqslant \| f \|^2 \qquad \text{for all } f \text{ in } \mathscr{H}, \text{ i.e., } VV^* \leqslant I - JJ^*. \quad (4)$$

Let $\mathscr{V}$ denote the set of all V in $\bar{\mathscr{B}}_1(M^\perp, \mathscr{H})$ satisfying (4). Note that if V is any operator of the form $V = (I - JJ^*)^{1/2}B$ where $B \in \bar{\mathscr{B}}_1(M^\perp, \bar{N})$ and $\bar{N}$ is the closure of $N \equiv \operatorname{Ran}(I - JJ^*)$, then

$$\| V^*f \|^2 = \| B^*(I - JJ^*)^{1/2} f \|^2 \leqslant \|(I - JJ^*)^{1/2} f \|^2$$

for all f in $\mathscr{H}$, so V is in $\mathscr{V}$. On the other hand, if V is in $\mathscr{V}$, define the operator T on $\mathscr{H}$, with dense domain the linear span of N and $N^\perp$, by

$$T = V^*(I - JJ^*)^{-1/2} P_{\bar{N}} .$$

Note using (4) that for f in $D(T)$,

$$\begin{aligned} \| Tf \|^2 &= \langle (I - JJ^*)^{-1/2} P_{\bar{N}} f, VV^*(I - JJ^*)^{-1/2} P_{\bar{N}} f \rangle \\ &\leqslant \langle (I - JJ^*)^{-1/2} P_{\bar{N}} f, (I - JJ^*)(I - JJ^*)^{-1/2} P_{\bar{N}} f \rangle \\ &\leqslant \| f \|^2 \end{aligned}$$

so $\| \bar{T} \| \leqslant 1$ where $\bar{T} \in \bar{\mathscr{B}}_1(\mathscr{H})$ denotes the closure of T. Let $B = (\bar{T})^*$. Then for f in N,

$$B^*(I - JJ^*)^{1/2} f = V^*(I - JJ^*)^{-1/2} P_{\bar{N}}(I - JJ^*)^{1/2} f = V^*f$$

and for f in $N^\perp$,

$$B^*(I - JJ^*)^{1/2} f = 0 = V^*f$$

so $V = (I - JJ^*)^{1/2}B$. Now $\| B \| \leqslant 1$ and $B^*P_{N^\perp} = 0$, so $BB^* \leqslant P_{\bar{N}}$. Also $BP_M = 0$, so $B^*B \leqslant P_{M^\perp}$, and therefore $B \in \bar{\mathscr{B}}_1(M^\perp, \bar{N})$. We have thus proven

LEMMA 2. *The contraction extensions $\tilde{J}$ of J with domain $\mathscr{H}$ are uniquely represented in the form $\tilde{J} = J + (I - JJ^*)^{1/2}B$, where B varies throughout $\bar{\mathscr{B}}_1(M^\perp, \bar{N})$.*

5. Extreme Semigroups and the Representation Theorems

Using Theorems P_1 and P_2, and Lemma 2, we may parametrize the set of operators $\tilde{L}$ which extend L and generate contraction semigroups, by the elements B of $\bar{\mathscr{B}}_1(M^\perp, \bar{N})$; the infinitesimal generator corresponding to B will be denoted L_B, with Cayley transform J_B and associated semigroup $\{U_B{}^t \mid t \geqslant 0\}$.

DEFINITION. Those infinitesimal generators L_B, and associated J_B and $\{U_B{}^t \mid t \geqslant 0\}$, which correspond to B's in $\mathscr{B}_E(M^{\perp}, \bar{N})$, will be called *extreme*.

Note that for $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda > 0$, $R(\lambda, B) = (\lambda I - L_B)^{-1}$ is the resolvent of L_B, and that $R(1, B) = (I + J_B)/2$. For $| \lambda - 1 | < 1$, a well-known argument [3, 5.8.4] yields

$$R(\lambda, B) = \sum_{n=1}^{\infty} (\lambda - 1)^{n-1} [R(1, B)]^n \tag{5}$$

so that $R(\lambda, B)$ is the limit of Gâteaux differentiable functions from $\mathscr{B}_1(M^{\perp}, \bar{N})$ into $\mathscr{B}(\mathscr{H})$, and therefore also has this property [3, 3.18.1]; therefore $[R(\lambda, B)]^m$ does also, for $m = 1, 2, \ldots$. By iteration, this argument extends to all $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda > 0$. From [3, 11.6.6],

$$U_B{}^t = \lim_{n \to \infty} [(n/t)\, R((n/t), B)]^n, \qquad t > 0$$

and therefore (using [3, 12.3.2 and 3.18.1]) $U_B{}^t$ is also Gâteaux differentiable on $\mathscr{B}_1(M^{\perp}, \bar{N})$ for fixed $t \geqslant 0$. Finally, from [4, Theorem 2.16] we see that for fixed $t \geqslant 0$, $U_B{}^t$ is continuous on $\mathscr{B}_1(M^{\perp}, \bar{N})$ and is therefore in $F(M^{\perp}, \bar{N})$. Using Lemma 1 and Theorems P_1 and P_2, we have proven

THEOREM 1. *If L is a densely defined dissipative operator on $\mathscr{H}$, every contraction semigroup $\{U_B{}^t \mid t \geqslant 0\}$ which is generated by an extension L_B of L can be represented in the form*

$$U_B{}^t = \int U_{B'}^t \, d\mu_B(B'), \qquad t \geqslant 0 \tag{6}$$

where μ_B is a regular Borel probability measure concentrated on the extreme semigroups (*and independent of t*).

The related problem where one seeks an infinitesimal generator which is the *restriction* of a given operator, is easily treated using the above analysis together with another result of Phillips [1],

THEOREM P_3. *The operator L on $\mathscr{H}$ is the infinitesimal generator of the contraction semigroup $\{U^t \mid t \geqslant 0\}$ if and only if L^* is the infinitesimal generator of the contraction semigroup $\{V^t \mid t \geqslant 0\}$ where $V^t \equiv (U^t)^*$.*

Using this theorem, the adjoint map produces a one-to-one inclusion reversing correspondence between the infinitesimal generators, of contraction semigroups, which are extensions of L and those which are restrictions of L^*. We label each of the latter with the $B \in \bar{\mathscr{B}}_1(M^\perp, \bar{N})$ of its adjoint, and call the infinitesimal generator, and its associated Cayley transform and semigroup, *extreme* if its adjoint is extreme by the previous definition. Since the adjoint map is continuous on $\bar{\mathscr{B}}_1(\mathscr{H})$, and the integral in (6) can be approximated by Riemann sums for the continuous integrand $U_{B'}^t$, Theorem 1 immediately implies

THEOREM 2. *If L is a densely defined dissipative operator on $\mathscr{H}$, every contraction semigroup $\{U_B{}^t \mid t \geqslant 0\}$ which is generated by a restriction L_B of L^* can be represented in the form*

$$U_B{}^t = \int U_{B'}^t \, d\mu_B(B'), \qquad t \geqslant 0 \tag{7}$$

where μ_B is a regular Borel probability measure concentrated on the extreme semigroups (and independent of t).

Now assume that $L = iH$ where H is symmetric and densely defined (which implies that L is dissipative). If for some conjugation K (i.e., antilinear isometric operator K on $\mathscr{H}$ such that $K^2 = I$) we have

$$KHK = H \tag{8}$$

(i.e., H "permutes" with K) then it is natural to look for infinitesimal generators $\tilde{L}$ satisfying either (a) $\tilde{L} \supseteq L$, or (b) $L \subseteq L^*$, and also

$$KLK = L^*. \tag{9}$$

In terms of Cayley transforms, (9) is equivalent to

$$KJK = J^* \tag{10}$$

and in terms of semigroups, (10) is equivalent to

$$KU^tK = (U^t)^*, \qquad t \geqslant 0. \tag{11}$$

From (10) it follows that K is an (antilinear) isometry of $\mathrm{Ran}(J)$ onto $\mathrm{Ran}(J^*)$. Since H is symmetric, J is an isometry of $M = J^*J\mathscr{H} = \mathrm{Ran}(J^*)$ onto $JJ^*\mathscr{H} = \mathrm{Ran}(J)$, and from (8), K maps $\mathrm{Ran}(J)$ isometrically onto $\mathrm{Ran}(J^*)$. Therefore using the terminology of the proofs of Lemmas 1 and 2, $\tilde{J}$ is of the form $\tilde{J} = J + B$ and $B(z) = UR(z) + zV = U + zV$. From the above remarks, K maps F isometrically onto G and $F^\perp = M^\perp \ominus F$ isometrically onto

$G^\perp = \bar{N} \ominus G$. Therefore V can be chosen to be an isometry of $F^\perp$ onto $G^\perp$ and for this choice $B[\exp(i\theta)]$, and $U^t_{B[\exp(i\theta)]}$, will be unitary. It is easy to choose V so as to also satisfy $KVK = V^*$, so that $KU^t_{B[\exp(i\theta)]}K = (U^t_{B[\exp(i\theta)]})^*$.

DEFINITION. A contraction semigroup which satisfies (11) will be said to be *reflective* (with respect to the conjugation K).

We summarize the above argument in a corollary:

COROLLARY 1 (resp. 2) to Theorem 1 (resp. 2). *If furthermore $L = iH$, where H is symmetric and permutes with a conjugation K, then if $U_B{}^t$ is reflective, the representation* (6) (*resp.* (7)) *of $U_B{}^t$ can be made in terms of reflective, unitary $U^t_{B'}$.*

We note that Corollaries 1 and 2 apply quite generally to Schrödinger equations in bounded regions.

The proof of Corollaries 1 and 2 can be simplified for H any densely defined symmetric operator with equal and finite deficiency indices; in such a case $\mathscr{B}_E(M^\perp, \bar{N})$ consists only of isometries of $M^\perp$ onto $\bar{N}$, so the extreme semigroups are unitary, and there is no need for a conjugation operator.

Finally, we wish to consider the connection between the above results, and that of Nelson in [2]. In [2], Nelson solves (1) for $\mathscr{H} = L_2(\mathbf{R}^n)$, and $L = i(-\nabla^2 + V)$, where V is the multiplication operator on $\mathscr{H}$ corresponding to any real function $V(x)$ continuous off a closed set S of capacity zero. Since the allowed $V(x)$ may be highly singular on S, a traditional approach to solving (1) would require the selection of some boundary condition at S. But Nelson seems to avoid such considerations by analytically continuing the *unique* solution of an associated diffusion equation. As suggested by [5], where a specific example ($V(x) = -(|c|/|x|)^2$, $n = 3$) is thoroughly examined, it seems reasonable to suspect that Nelson's analytic continuation technique is in general merely a method of averaging over time reversible evolutions in the sense of (6) or (7). This indeed follows immediately from our Corollary 1 since the hypotheses are easily seen to be satisfied with K the complex conjugation operator on $L_2(\mathbf{R}^n)$. It is suggestive that the averaging schemes (6) and (7) are superimposed in [2] on the stochastic aspects of Feynman and Wiener integrals for domains with boundary.

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