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Conjugacies for Tiling Dynamical Systems

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Abstract: We consider tiling dynamical systems and topological conjugacies between them. We prove that the criterion of being of finite type is invariant under topological conjugacy. For substitution tiling systems under rather general conditions, including the Penrose and pinwheel systems, we show that substitutions are invertible and that conjugacies are generalized sliding block codes.

1. Notation and Main Results

We begin with a definition of tiling dynamical systems, in sufficient generality for this work. Let $\mathcal{A}$ be a nonempty finite collection of compact connected sets in the Euclidean space $\mathbb{E}^d$, sets with dense interior and boundary of zero volume. Let $X(\mathcal{A})$ be the set of all tilings of $\mathbb{E}^d$ by congruent copies, which we call tiles, of the elements of the "alphabet" $\mathcal{A}$. We assume $X(\mathcal{A})$ is nonempty, which is automatic for the special class of substitution tiling systems on which we will concentrate below. We label the "types" of tiles by the elements of $\mathcal{A}$. We endow $X(\mathcal{A})$ with the metric

$$m[x, y] \equiv \sup_{n\geq 1} \frac{1}{n} m_H[B_n \cap \partial x, B_n \cap \partial y], \tag{1}$$

where B_n denotes the open ball of radius n centered at the origin $\mathbf{O}$ of $\mathbb{E}^d$, and ∂x the union of the boundaries of all tiles in x. (A ball centered at $\mathbf{a}$ is denoted $B_n(\mathbf{a})$.) The Hausdorff metric m_H is defined as follows. Given two compact subsets P and Q of $\mathbb{E}^d$, $m_H[P, Q] = \max\{\tilde{m}(P, Q), \tilde{m}(Q, P)\}$, where

$$\tilde{m}(P, Q) = \sup_{p\in P} \inf_{q\in Q} \|p - q\|, \tag{2}$$

with $\|w\|$ denoting the usual Euclidean norm of w.

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Under the metric m two tilings are close if they agree, up to a small Euclidean motion, on a large ball centered at the origin. The converse is also true for tiling systems with finite local complexity (as defined below): closeness implies agreement, up to small Euclidean motion, on a large ball centered at the origin [see RaS1]. Although the metric m depends on the location of the origin, the topology induced by m is Euclidean invariant. A sequence of tilings converges in the metric m if and only if its restriction to every compact subset of $\mathbb{E}^d$ converges in m_H. It is not hard to show [RW] that $X(\mathcal{A})$ is compact and that the natural action of the connected Euclidean group $\mathcal{G}_E$ on $X(\mathcal{A})$, $(g, x) \in \mathcal{G}_E \times X(\mathcal{A}) \mapsto g[x] \in X(\mathcal{A})$, is continuous.

To include certain examples it is useful to generalize the above setup, to use what is sometimes called "colored tiles". To make the generalization we assign a "color" from some finite set to each element of $\mathcal{A}$, represented on each tile by a "color marking", a line segment in the interior of the tile, of different length for different colors. We then redefine ∂x as the union of the tile boundaries and color markings in the tiling x.

Definition 1. *A tiling dynamical system is the action of $\mathcal{G}_E$ on a closed, $\mathcal{G}_E$-invariant subset of $X(\mathcal{A})$.*

We emphasize the close connection between such dynamical systems and subshifts. A subshift with $\mathbb{Z}^d$-action is the natural action of $\mathbb{Z}^d$ on a compact, $\mathbb{Z}^d$-invariant subset X of $\mathcal{B}^{\mathbb{Z}^d}$, for some nonempty finite set $\mathcal{B}$. If we associate with each element of $\mathcal{B}$ a "colored" unit cube in $\mathbb{E}^d$, the face-to-face tilings of $\mathbb{E}^d$ by those arrays of such cubes corresponding to the subshift X gives a tiling dynamical system which is basically the suspension of the subshift X (but with rotations of the entire tiling also permitted).

A significant difference between subshifts and tiling dynamical systems is that for (nontrivial) subshifts the group acts on a Cantor set, while the space is typically connected for interesting tiling systems. In fact, the spaces for different tiling systems need not be homeomorphic.

A major objective in dynamics is the classification of interesting subclasses up to topological conjugacy. For the class of subshifts a central theorem, due to Curtis, Lyndon and Hedlund, shows that a topological conjugacy can be represented by a sliding block code (see [LM]). For tiling dynamical systems there is a natural analogue of such a representation for which we use the same term. (Such maps are called "local" in [P] and are closely related to mutual local derivability [BSJ].)

Definition 2. *A topological conjugacy $\psi : X_{\mathcal{A}} \mapsto X_{\mathcal{A}'}$ between tiling systems is a sliding block code if for every $n' > 0$ there is $n > 0$ such that for every $x, y \in X_{\mathcal{A}}$ such that $B_n \cap \partial x = B_n \cap \partial y$ we have $B_{n'} \cap \partial(\psi x) = B_{n'} \cap \partial(\psi y)$.*

Our first result is:

Theorem 1. *Within the subclass of substitution tiling systems with invertible substitution, every topological conjugacy is a sliding block code.*

Before defining the subclass of "substitution" tiling systems in general we present some relevant examples.

A "Penrose" tiling of the plane, Fig. 1, can be made as follows. Consider the 4 (colored) tiles of Fig. 2. Divide each tile (also called a "tile of level 0") into 2 or 3 pieces as in Fig. 2 and rescale by a linear factor of the golden mean $\tau = (1 + \sqrt{5})/2$ so that each piece is the same size as the original. This yields 4 collections of tiles that we call "tiles of level 1". Subdividing each of these tiles and rescaling gives 4 collections of tiles that we call tiles of level 2. Repeating the process n times gives tiles of level n. A Penrose

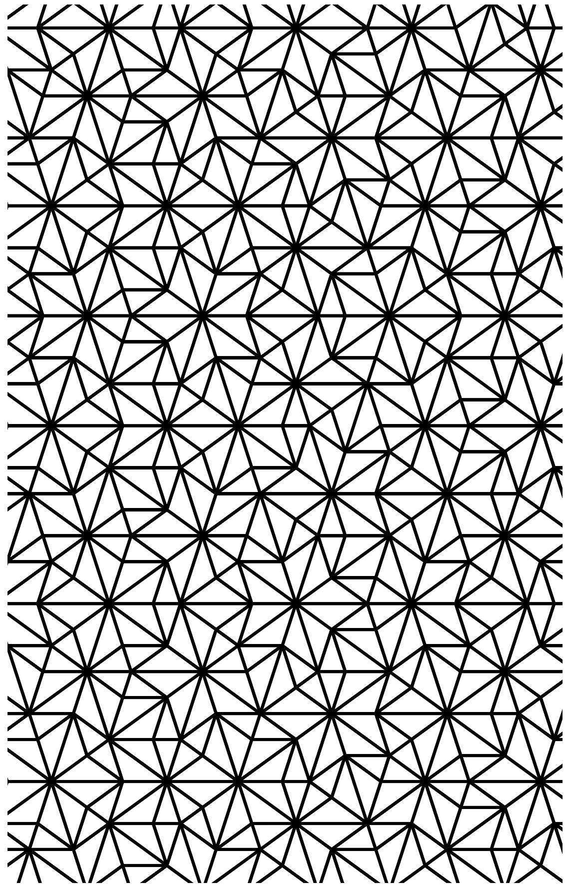

Fig. 1. A Penrose tiling

tiling is a tiling of the plane with the property that every finite subcollection of tiles is congruent to a subset of a tile of some level. A Penrose tiling has only 4 types of tiles, each appearing in 10 different orientations.

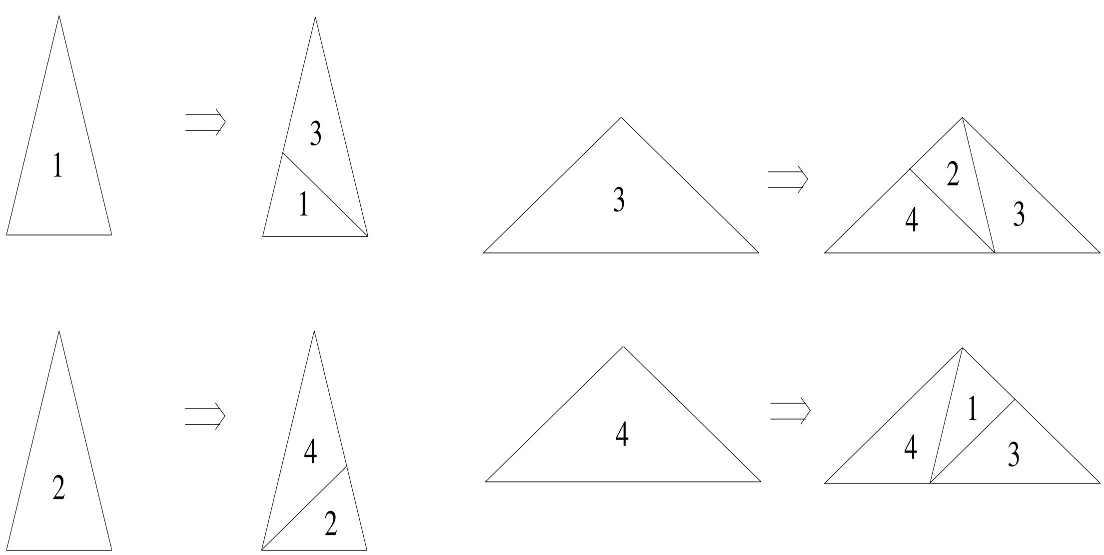

Fig. 2. The Penrose substitution

A "pinwheel" tiling of the plane, Fig. 3, uses two basic tiles: a 1-2-$\sqrt{5}$ right triangle and its mirror image, as shown in Fig. 4 with their substitution rule [R1]. Notice that at the center of each tile of level 1 there is a tile of level 0 similar to the level 1 tile but rotated by an angle $\alpha = \tan^{-1}(1/2)$. Thus the center tile of a tile of level n is rotated by $n\alpha$ relative to the tile. Since α is an irrational multiple of π, we see, using the fact that within a tile of level 2 there is a tile of level 0 similar and parallel to the level 2 tile, that this rotation never ends, and each tiling contains tiles in infinitely many distinct orientations.

More generally, for any integers $m < n$ we consider the "(m, n)-pinwheel" tilings defined (for $m = 3,\ n = 4$) by the substitution of Fig. 5, whose tiles are m-n-$\sqrt{m^2 + n^2}$ right triangles. Like the ordinary pinwheel, such variant pinwheel tilings also necessarily have tiles in infinitely many distinct orientations.

It is easy to construct explicit examples of Penrose and pinwheel tilings. Pick a tile to include the origin of the plane. Embed this tile in a tile of level 1 (there are several ways to do this). Embed that tile of level 1 in a tile of level 2, embed that in a tile of

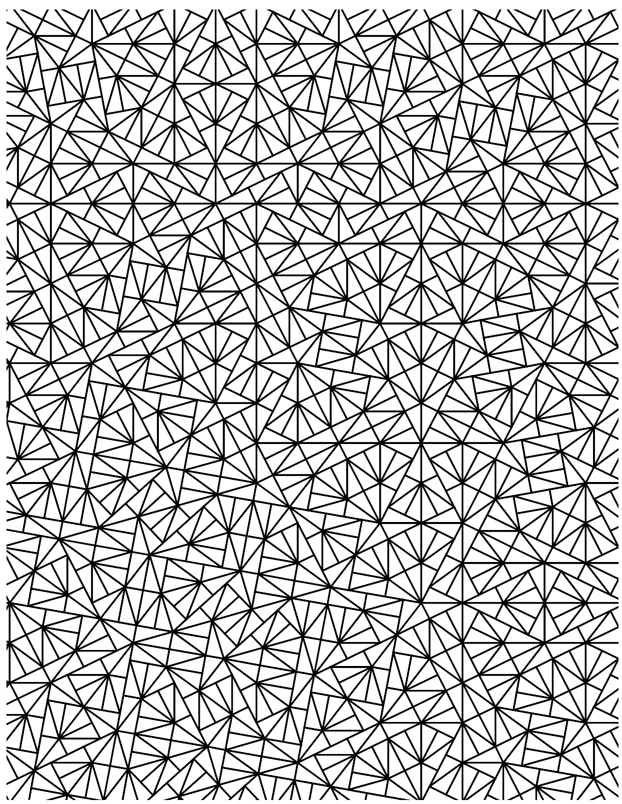

Fig. 3. A pinwheel tiling

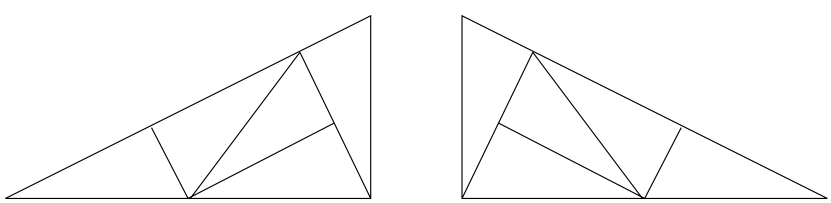

Fig. 4. The pinwheel substitution

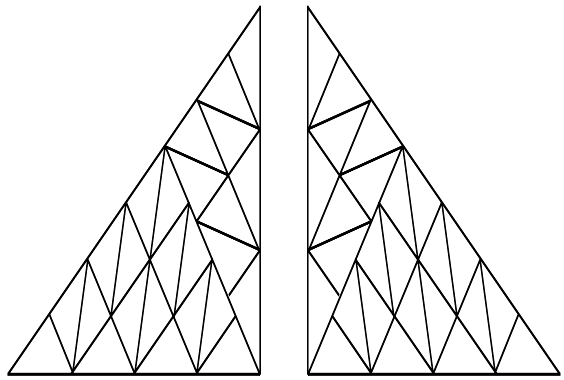

Fig. 5. The substitution for (3,4)-pinwheel tilings

level 3, and so on. The union of these tiles of all levels will cover an infinite region, typically—though not necessarily—the entire plane.

In order to generalize from the above examples we need some further notation. A "patch" is a (finite or infinite) subset of a tiling $x \in X(\mathcal{A})$; the set of all finite patches for $\mathcal{A}$ will be denoted $W_{\mathcal{A}}$. A "substitution function" ϕ is a map from $W_{\mathcal{A}}$ to itself defined by a decomposition of each tile type, stretched linearly about the origin by a factor $\lambda_\phi > 1$, into congruent copies of the original tiles. (Recall the Penrose and pinwheel examples.) We assume:

i) For each $k > 0$ there are only a finite number of possible patches, up to Euclidean motion, obtained by taking a ball of radius k around any point inside a tile T of level n, $\phi^n(T)$, where T and n are arbitrary. (This is usually called "finite local complexity".)
ii) For each tile $T \in \mathcal{A}$, $\phi(T)$ contains at least one tile of each type. (This is usually called "primitivity".)
iii) For every tile $T \in \mathcal{A}$ there is $n_T \geq 1$ such that $\phi^{n_T}(T)$ contains a tile of the same type as T and parallel to it.

Condition i) is highly significant, the remaining conditions much less so. In interesting cases condition ii) can usually be obtained by replacing the substitution by a power of itself; note that this does not affect the tiling dynamical system at all. Condition iii) is related to the existence of a fixed point of the substitution; we know of no interesting examples of systems not satisfying this condition.

Definition 3. *For a given alphabet $\mathcal{A}$ of (possibly colored) tiles, and a substitution function ϕ, the "substitution tiling system" is the compact subspace $X_\phi \subset X(\mathcal{A})$, invariant under $\mathcal{G}_E$, of those tilings x such that every finite subpatch of x is congruent to a subpatch of $\phi^n(T)$ for some $n > 0$ and $T \in \mathcal{A}$. The map ϕ extends naturally to a continuous map (again denoted ϕ) from X_ϕ into itself.*

There are two natural relaxations of this definition of substitution tiling systems. For any tiling system X and any positive constant λ, let λX denote the system of tilings obtained by rescaling each tiling in X by λ. If X_ϕ is a substitution tiling system, then X_ϕ and $\lambda_\phi X_\phi$ are topologically conjugate, via a sliding block code that associates tiles (of level 0) of tilings in $\lambda_\phi X_\phi$ with tiles of level 1 of tilings in X_ϕ.

Definition 4. *If a tiling system X has the property that, for some $\lambda > 1$, λX and X are topologically conjugate via a sliding block code, then X is a "pseudo-substitution tiling* *system", and the map $X \mapsto X$ obtained by first rescaling by λ and then applying the conjugacy is called a "pseudo-substitution". If X and λX are topologically conjugate (not necessarily via a sliding block code), then X is a "quasi-substitution tiling system".*

In the literature, pseudo-substitutions are sometimes called "improper substitutions" or "substitutions with amalgamation", and their fixed points are called "pseudo-self-similar tilings". In 2 dimensions, and with some additional assumptions, the categories of substitution tiling systems and of pseudo-substitution tiling systems are essentially identical, thanks to a construction of Priebe and Solomyak [PS] that converts a pseudo-self-similar tiling into a self-similar tiling. It is generally believed that this construction can be generalized to higher dimensions, dropping some of the restrictive assumptions, in which case properties of substitution tiling systems, such as Theorem 1, can be expected to apply to pseudo-substitution tiling systems. However, the following example (see also [P]) shows that the conclusions of Theorem 1 do *not* apply to quasi-substitution tiling systems.

Following [RS2], we consider suspensions of the 1-dimensional Fibonacci substitution subshift. That subshift is defined by the alphabet $\mathcal{B} = \{0, 1\}$ and the substitution of 0 by 1 and of 1 by the word 0 1. One can make a family of suspensions of this subshift by replacing 0 and 1 by marked closed intervals of any positive lengths. One gets a substitution tiling system if 0 is associated with a segment T_0 of length $|T_0| = 1$ and 1 is associated with a segment T_1 of length the golden mean $\tau = 1 + \sqrt{5}/2$, $|T_1| = \tau$. When $|T_1|/|T_0| \neq \tau$, the resulting tiling system is merely a quasi-substitution tiling system [CS]. It was proven in [RS2] that two different Fibonacci tiling systems, defined by T_0, T_1 and T_0', T_1', are topologically conjugate if $|T_0| + \tau|T_1| = |T_0'| + \tau|T_1'|$, but that *such a conjugacy cannot be a sliding block code.*

Definitions 3 and 4 yield spaces of tilings in which the tiles appear in all orientations, although tiles appear in at most countably many different orientations within any *fixed* tiling. Any Penrose tiling, for example, has tiles in only 10 distinct orientations, but the space of *all* Penrose tilings contains all rotated versions of any tiling.

For each tiling x and each $r > 0$, consider the set of Euclidean motions g for which x and gx agree exactly on a ball of radius r around the origin. The subgroup of $SO(d)$ generated by the rotational parts of the g's is denoted $\mathcal{G}_{\mathrm{RO}}(r, x)$ and is called the *relative orientation group* of x. In [RS1] it was shown that $\mathcal{G}_{\mathrm{RO}}(r, x)$ is independent of r (and we henceforth write it as $\mathcal{G}_{\mathrm{RO}}(x)$), and that the groups for different tilings $x \in X_\phi$ are related by inner automorphism of $SO(d)$. There are no inner automorphisms of $SO(2)$, so in 2 dimensions the group is exactly the same for all tilings $x \in X_\phi$.

For the Penrose tiling, $\mathcal{G}_{\mathrm{RO}} = \mathbb{Z}_{10}$ is the set of rotations by multiples of $2\pi/10$. For the pinwheel tiling, $\mathcal{G}_{\mathrm{RO}}$ is generated by rotations of $\pi/2$ and $2\tan^{-1}(1/2)$. For the (m, n)-pinwheel tiling, the group is generated by rotations by $\pi/2$ and $2\tan^{-1}(m/n)$.

From Theorem 1 we deduce that the relative orientation group is a conjugacy invariant:

Corollary 1. *If $\psi : X_1 \to X_2$ is a topological conjugacy between substitution tiling systems with invertible substitution, and if $x \in X_1$, then $\mathcal{G}_{\mathrm{RO}}(\psi(x)) = \mathcal{G}_{\mathrm{RO}}(x)$.*

As an application, consider the (1,2)-pinwheel and (3,4)-pinwheel tilings. These have stretching factors $\sqrt{5}$ and 5, and so cannot be distinguished by the homeomorphism invariant of [ORS]. Moreover, their relative orientation groups are each isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}$ as abstract groups. However, these groups are different as subsets of $SO(2)$ (one is an index-2 subgroup of the other), so the two tiling systems cannot be topologically conjugate.

For each tiling $x \in X_\phi$, consider the closure of the orbit of x under translations. Let $\mathcal{G}_{rel}(x)$ be the maximal subgroup of $\mathcal{G}_E$ that maps this orbit closure to itself. It is not hard to see that $\mathcal{G}_{rel}(x)$ is the smallest subgroup of $\mathcal{G}_E$ that contains the closure of $\mathcal{G}_{\text{RO}}(x)$ in $SO(d)$ and contains all translations. If the translation orbit of x is dense in X_ϕ, as with the pinwheel and its variants, then $\mathcal{G}_{rel}(x) = \mathcal{G}_E$. We should add that for tiling systems in which tiles only appear in finitely many orientations in any tiling, it can be convenient to consider the dynamical system of translations on the orbit closure of such a tiling. Although our convention is to allow all of the Euclidean group to act on tilings it is easy to obtain corresponding information about these more limited dynamical systems from our results.

Our second result concerns what is often called the "recognizability" or "unique composition property" of the substitution:

Theorem 2. *If for a substitution function ϕ there is some tiling x in X_ϕ not fixed by any translation, for which the orbit under translation is dense in X_ϕ, then the extension of ϕ to X_ϕ is a homeomorphism.*

Theorem 2 complements a result of Solomyak [S], which dealt with tiling systems that had finite relative orientation groups. Solomyak's result was itself a generalization of Mossé's work on 1-dimensional subshifts [Mos]. For tilings in 3 or more dimensions there is an additional case, where the relative orientation group is infinite but not dense in $SO(d)$. We have constructed a pseudo-substitution tiling system in 3 dimensions, with $\mathcal{G}_{RO}$ a dense subgroup of $SO(2)$, for which the pseudo-substitution is *not* a homeomorphism. However, the recognizability of true substitutions in 3 or more dimensions remains open.

Our third result concerns "finite type". If X is a tiling space and $r > 0$, let X_r be the set of tilings for which every patch of radius r also appears in some tiling in X. If $r_1 > r_2$, then $X_{r_1} \subseteq X_{r_2}$, and it is easy to show that $\cap_r X_r = X$. If $X = X_r$ for some finite r, then we say that X is of *finite type*. Roughly speaking, this means that the patterns in X are defined by local conditions, whose range is at most $2r$. For subshifts, it is well known that being of finite type is an invariant of topological conjugacy. (See [RS2] for an explicit proof of this folk theorem.) We extend this to tiling systems:

Theorem 3. *Let X, Y be topologically conjugate tiling systems, each of finite local complexity. X is of finite type if and only if Y is of finite type.*

2. Proofs of Theorems 1 and 2, and Some Related Results

We begin with the proof of Theorem 2. We abbreviate X_ϕ by X and call a patch admissible if there is a tiling $x \in X$ containing it. We assume ϕ has a fixed point in X, not fixed by any translation, whose orbit under translations is dense in X. (The existence of a periodic point for ϕ follows from iii), and we are free to replace ϕ with a higher power.) Let $H : \mathcal{G}_E \mapsto \mathcal{G}_E$ be defined by $\phi(gx) = H(g)\phi(x)$.

The following four lemmas are proved with standard arguments, as sketched below.

Lemma 1. *The extension $\phi : X \mapsto X$ is surjective.*

Sketch of proof. Since X is compact, $\phi(X)$ is a closed subset of X. To see that it is dense, note that any admissible patch is a subset of some tile of level n. Thus, for any tiling $x \in X$, and any $r > 0$, $B_r^x = B_r^{\phi(y)}$ for some tiling y. □

Lemma 2. *There is a constant $C > 0$ such that for every $r > 0$ and for every pair of admissible patches P, P' with supp$(P) \subset B_r$ and $B_{Cr} \subset$ supp(P') there exists $g \in \mathcal{G}_E$ such that $gP \subset P'$.*

Sketch of proof. By finite local complexity and primitivity, there exists N such that for every $x \in X$ and for every prototile T, the patch $\phi^N(T)$ contains a congruent copy of B^x_m. Take C greater than λ^{N+1}/m times the maximum diameter of a tile. Suppose r, P, P' are as in the statement of the lemma. Let n be the least integer such that $r\lambda^{-n} \leq m$. Then P' contains a tile of level $N+n$, and every tile of level $N+n$ contains a congruent copy of P. □

Lemma 3. *If $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \mathbb{E}^d$ and $\mathbf{t} = \sum_{i=1}^k a_i \mathbf{b}_i$ with $a_i \in \mathbb{N}$, then there exist $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_\ell$ such that $\mathbf{t}_0 = 0$, $\mathbf{t}_\ell = \mathbf{t}$, and for each $j = 1, \ldots, \ell$,*

$$\mathbf{t}_j - \mathbf{t}_{j-1} \in \{\pm \mathbf{b}_i : i = 1, \ldots, k\}, \tag{3}$$

and $\mathbf{t}_j$ lies within $(k/2) \max_{1 \leq i \leq k} \|\mathbf{b}_i\|$ of the straight-line path from $\mathbf{O}$ to $\mathbf{t}$.

Sketch of proof. Each point along the straight-line path from $\mathbf{O}$ to $\mathbf{t}$ is a linear combination of the $\mathbf{b}_i$'s with real coefficients. Round these coefficients to the nearest integer to get the sequence of $\mathbf{t}_j$'s. □

Lemma 4. *If $\mathcal{G}_{rel}(y) = \mathcal{G}_E$ there is a constant D such that if P, P' are admissible patches in y with supp$(P) \subset B_r$ and $B_{Dr} \subset$ supp(P') then there exist $\alpha_1, \ldots, \alpha_n \in SO(d)$ and $\mathbf{t}_1, \ldots, \mathbf{t}_n \in \mathbb{R}^d$ such that $\alpha_i P + \mathbf{t}_i \subset P'$, $i = 1, 2, \ldots, n$, and such that no proper subspace of $\mathbb{E}^d$ is invariant under all the α_i.*

Sketch of proof. This is similar to Lemma 2. One can prove it first when P is a tile and then extend the result to larger patches by inverting the substitution. □

Let $M = \max\{\text{diam}(T) : T \in \mathcal{A}\}$. Let us say that a patch P has *period* $g \in \mathcal{G}_E$ if $P \cup gP$ is a patch, i.e., if P and gP agree where their supports overlap (we do not require that they actually overlap). Alternatively, P has period $g \in \mathcal{G}_E$ if and only if whenever $T \in P$ is such that gT° intersects supp(P) we also have $gT \in P$. Of course any subpatch of a patch of period g has period g.

Lemma 5. *If $\{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\} \subset \mathbb{E}^d$ is a basis for a lattice $\mathcal{L}$ and P is a patch having all periods in $\mathcal{L}$ and additional translational period $\mathbf{b}_k$ such that $B_r \subset$ supp(P), where*

$$r > (d+1) \max_{1 \leq i \leq k} \|\mathbf{b}_i\| + 4M, \tag{4}$$

then $\langle \mathbf{b}_1, \ldots, \mathbf{b}_k \rangle$ is a lattice of periods for $B^P_{r/2}$.

Proof. Suppose $T \in B^P_{r/2}$ and $T^\circ + \mathbf{t}$ intersects supp$(B^P_{r/2})$, where $\mathbf{t} = a_1\mathbf{b}_1 + \cdots + a_k\mathbf{b}_k$ with each $a_i \in \mathbb{N}$. Let $\mathbf{b} \in T \cap B_{r/2}$. Then $\mathbf{b} + \mathbf{t} \in B_{r/2+2M}$. Let $\mathbf{t}_0, \ldots, \mathbf{t}_\ell$ be as in Lemma 3. The straight-line path from $\mathbf{b}$ to $\mathbf{b} + \mathbf{t}$ lies in $B_{r/2+2M}$ so each $\mathbf{b} + \mathbf{t}_j$ is in B_r. Thus, for each j we have $(T^\circ + \mathbf{t}_j) \cap P \neq \emptyset$ and by finite induction $T + \mathbf{t}_j \in P$. It follows that $T + \mathbf{t} \in B^P_{r/2}$. □

Lemma 6. *If $k < d$ and $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is a basis for a lattice $\mathcal{L}$ and P is an admissible patch having all periods in $\mathcal{L}$ such that $B_r \subset supp(P)$, where*

$$r > D((d+1) \max_{1 \le i \le k} \|\mathbf{b}_i\| + 4M), \tag{5}$$

then there exists $\mathbf{b}_{k+1} \in \mathbb{E}^d \setminus span(\mathcal{L})$ with $\|\mathbf{b}_{k+1}\| \le \max_{1 \le i \le k} \|\mathbf{b}_i\|$ and such that $\langle \mathbf{b}_1, \dots, \mathbf{b}_{k+1} \rangle$ is a lattice of periods for $B^P_{r/3D}$.

Proof. Let $P' = B^P_{r/D-M}$. By Lemma 4, there exist $\alpha \in SO(d)$ and $\mathbf{t} \in \mathbb{R}^d$ such that $\alpha P' + \mathbf{t} \subset P$ and $\alpha\ \mathrm{span}(\mathcal{L}) \ne \mathrm{span}(\mathcal{L})$. It follows that P' has all periods in $\mathcal{L}$ as well as all periods in $\alpha^{-1}\mathcal{L}$. Let $\mathbf{b}_{k+1} \in \alpha^{-1}\{\mathbf{b}_1, \dots, \mathbf{b}_k\} \setminus \mathrm{span}(\mathcal{L})$. By Lemma 5, $\langle \mathbf{b}_1, \dots, \mathbf{b}_{k+1} \rangle$ is a lattice of periods for $B^P_{r/2D-M/2}$. Since $r/2D - M/2 > r/3D$, the proof is complete. □

We next show that a large patch cannot have periods which are small relative to the size of the patch.

Proposition 1. *There is a constant $K > 0$ such that if P is an admissible patch whose support contains a ball of radius r then every non-identity period g of P satisfies $\|g\mathbf{b} - \mathbf{b}\| > Kr$ for some $\mathbf{b} \in supp(P)$.*

Proof. It suffices to prove there is a K which satisfies the conclusion for all sufficiently large r. Recall the notation of m as the inner radius. No tiling in X has a translational period of magnitude less than m. Let $K > 0$ be less than each of the following:

a) $4C\lambda(\lambda+1)(3D)^{d-1}(d + 4M/m)^{-1}$,
b) $\inf\{\|\alpha - I\|_{\mathrm{operator}} : I \ne \alpha \in SO(d)$ and α fixes some element of $X\}$,
c) $\frac{1}{4C}$.

Suppose there exist $r > 4MC$ and a patch P having a non-identity period $g \in \mathcal{G}_E$ with $\|g\mathbf{b} - \mathbf{b}\| \le Kr$ for all $\mathbf{b} \in \mathrm{supp}(P)$, and such that $B_r \subset \mathrm{supp}(P)$. Let $x \in X$ be a fixed point for ϕ, let $P' = B^x_{r/C-M}$, and let $h \in \mathcal{G}_E$ be such that $hP' \subset P$. Then $h^{-1}gh$ is a period for P' and

$$\|h^{-1}gh\mathbf{b} - \mathbf{b}\| \le Kr, \quad \text{for all } \mathbf{b} \in P'. \tag{6}$$

Since $\phi(P') \supset P'$ and $H(h^{-1}gh)$ is a period for $\phi(P')$, $H(h^{-1}gh)(h^{-1}gh)^{-1}$ is a period for $P' \cap (h^{-1}gh)^{-1}P'$. We have

$$B^x_{r(\frac{1}{C}-K)-M} \subset P' \cap (h^{-1}gh)^{-1}P' \tag{7}$$

and

$$\|H(h^{-1}gh)(h^{-1}gh)^{-1}\mathbf{b} - \mathbf{b}\| \le (\lambda+1)Kr, \text{ for all } \mathbf{b} \in \mathrm{supp}(B^x_{r(\frac{1}{C}-K)-M}). \tag{8}$$

Now $H(h^{-1}gh)(h^{-1}gh)^{-1}$ is a translation, say by $\mathbf{b}_1 \in \mathbb{R}^d$, and if $\mathbf{b}_1 = 0$ then $h^{-1}gh \in SO(d)$ and $h^{-1}ghb = b$. This last is impossible due to our choice of K, hence

$$0 < \|x_1\| \le (\lambda+1)Kr. \tag{9}$$

An application of Lemma 5 followed by $d-1$ applications of Lemma 6 (we will see in a moment that r is large enough for this) yields a d-dimensional lattice $\mathcal{L} = \langle \mathbf{b}_1, \ldots, \mathbf{b}_d \rangle$ of periods for $P' = B^x_{r'}$, where

$$\|\mathbf{b}_i\| \leq (\lambda + 1)Kr, \qquad i = 1, \ldots, d, \tag{10}$$

and

$$\begin{aligned} r' &= \frac{1}{2(3D)^{d-1}} r\left(\frac{1}{C} - K\right) - M \\ &\geq \frac{r}{4C(3D)^{d-1}} \\ &\geq (d + 4M/m)\lambda(\lambda + 1)Kr \\ &\geq (d + 4M/m)\lambda \max_{1 \leq i \leq d} \|x_i\|. \end{aligned} \tag{11}$$

Thus $r' > \lambda(\sum_{i=1}^d \|x_i\| + M)$, so the fundamental domain $F = \{t_1\mathbf{b}_1 + \cdots + t_d\mathbf{b}_d : t_i \in [0, 1]\}$ for $\mathcal{L}$ is such that $\phi(F^x) \subset B^x_{r'}$. This implies that every tile in $\phi(F^x)$ is a translate by an element of $\mathcal{L}$ of a tile in F^x, and it follows that all tiles in x are translates of tiles in F^x, contradicting that x should have tiles in infinitely many different orientations. □

For a tile T and $n \geq 0$ let $\mathcal{P}_n(T)$ be the set of admissible patches P for which $\phi^n(T) \subset \phi^n(P)$ and $\phi^n(T) \not\subset \phi^n(P')$ for any proper subpatch P' of P. Then each $\mathcal{P}_n(T)$ is finite and $\{T\} = \mathcal{P}_0(T) \subset \mathcal{P}_1(T) \subset \cdots$.

Lemma 7. *For each tile T there is a positive integer N_T such that $\mathcal{P}_{N_T}(T) = \mathcal{P}_{N_T+1}(T) = \cdots$.*

Proof. Set $\mathcal{P}(T) = \cup_{n\geq 0}\mathcal{P}_n(T)$ and let $r > 0$, $\mathbf{y} \in \mathbb{E}^d$ such that $B_r(\mathbf{y}) \subset T$. By finite local complexity, $\mathcal{P}(T)$ has only finitely many patches up to rigid motion, since every $P \in \mathcal{P}(T)$ is of the form $(T^\circ)^x$ for some tiling $x \in X$.

If $P, gP \in \mathcal{P}_n(T)$ for some $g \in \mathcal{G}_E$ with $\|g\mathbf{b} - \mathbf{b}\| < Kr$ for all $\mathbf{b} \in P$ then $H^n(g)$ is a period for $\phi^n(P)$ which violates Proposition 1. Thus for each patch $P \in \mathcal{P}(T)$ the set of $g \in \mathcal{G}_E$ for which $gP \in \mathcal{P}(T)$ is discrete and bounded, hence finite. It follows that $\mathcal{P}(T)$ is finite, which is equivalent to the desired result. □

Lemma 8. *Suppose $\mathcal{P}_n(T) = \mathcal{P}_{n+1}(T)$. If $x \in X$ is such that $\phi^{n+1}(T) \subset \phi(x)$ then $\phi^n(T) \subset x$.*

Proof. Let $x' \in X$ be such that $\phi^n(x') = x$. Then $\phi^{n+1}(T) \subset \phi(x) = \phi^{n+1}(x')$, hence there exists $P \in \mathcal{P}_{n+1}(T)$ such that $P \subset x'$. Since $P \in \mathcal{P}_n(T)$ we have $\phi^n(T) \subset \phi^n(P) \subset \phi^n(x') = x$. □

Proof of Theorem 2. Set $N = \max\{N_T : T \in \mathcal{A}\}$. Let $x \in X$, and let $x_1, x_2 \in X$ be any tilings such that

$$\phi(x_1) = \phi^{N+1}(x_2) = x. \tag{12}$$

We only need to show that $x_1 = \phi^N(x_2)$.

Let $T \in x_2$ be any tile and let $g \in \mathcal{G}_E$ be such that gT is a tile in $\mathcal{A}$. Then

$$\phi^{N+1}(gT) \subset \phi^{N+1}(gx_2) = \phi(H^N(g)x_1). \tag{13}$$

By Lemma 8, $\phi^N(gT) \subset H^N(g)x_1$, and hence $\phi^N(T) \subset x_1$. □

Remarks. The preceding arguments show that Proposition 1 is tantamount to recognizability, even for substitutions that do not satisfy hypothesis iii) and hence do not have a fixed point. This equivalence will be used in the proof of Propositions 3 and 4, below. The existence of a fixed point and the fact that $\mathcal{G}_{rel} = \mathcal{G}_E$ were used to prove Proposition 1. We believe the conclusion is false without the latter assumption.

We now begin the proof of Theorem 1. For $i = 1, 2$, let ϕ_i be a substitution on alphabet $\mathcal{A}_i$ with linear scaling factor λ_i such that $\phi_i : X_{\phi_i} \mapsto X_{\phi_i}$ is a homeomorphism. Write X_i for X_{ϕ_i} and suppose $\psi : (X_1, \mathcal{G}_E) \mapsto (X_2, \mathcal{G}_E)$ is a topological conjugacy.

Notation. For $r > 0$, $\mathbf{a} \in \mathbb{E}^d$ and a tiling y, $B_r(\mathbf{a})^y$ denotes the patch of y consisting of all tiles in y that intersect the open ball of radius r about $\mathbf{a}$. We abbreviate $B_r(\mathbf{O})^y$ as B_r^y. Pick an "inner radius" m such that every tile in $\mathcal{A}_1 \cup \mathcal{A}_2$ contains an open ball of radius m. For patch-valued functions P, Q on X_1 we say P determines Q (or Q is determined by P) if whenever $x, y \in X_1$ and $P(x) = P(y)$ we also have $Q(x) = Q(y)$.

It follows from the fact that $\phi_i : X_i \mapsto X_i$ is a homeomorphism that there is a "recognizability radius" $D_i > 0$ such that for $x \in X_i$ the patch $B_{D_i}^x$ determines the patch consisting of tiles containing the origin in $\phi_i^{-1}(x)$.

Lemma 9. *There is a constant $\rho > 0$ such that if $n \in \mathbb{N}$ and $r > \lambda_2^n \rho$ then for $y \in X_2$ the patch $B_{r/(2\lambda_2^n)}^{\phi_2^{-n}(y)}$ is determined by the patch B_r^y.*

Proof. Take $\rho = 2D_2/(\lambda_2 - 1)$. A patch of radius r in y determines a patch of radius $(r - D_2)/\lambda_2$ in $\phi_2^{-1}(y)$, of radius $[(r - D_2)/\lambda_2 - D_2]/\lambda_2$ in $\phi_2^{-2}(y)$, and $r/\lambda_2^n - D_2(\lambda_2^{-1} + \lambda_2^{-2} + \cdots + \lambda_2^{-n})$ in $\phi_2^{-n}(y)$. This last radius is greater than $r\lambda_2^{-n} - D_2/(\lambda_2 - 1)$, which in turn is at least $r/2\lambda_2^n$.

Any element $g \in \mathcal{G}_E$ can be written uniquely as the composition of a rotation and a translation, i.e., there exist unique $\alpha \in SO(d)$ and $\mathbf{s} \in \mathbb{R}^d$ such that

$$g\mathbf{a} = \alpha\mathbf{a} + \mathbf{s} \text{ for all } \mathbf{a} \in \mathbb{E}^d \tag{14}$$

and we set

$$\ell(g) = \|\alpha - I\|_{\text{operator}} + \|\mathbf{s}\|. \tag{15}$$

Notation. For patch-valued functions P, Q on X_1 the phrase "P determines Q up to motion by some $g \in \mathcal{G}_E$ with $\ell(g) \le \eta$" means that if $x, y \in X_1$ are such that $P(x) = P(y)$ then there exists $g \in \mathcal{G}_E$ with $\ell(g) \le \eta$ such that $Q(x) = gQ(y)$.

Lemma 10. *There exist a constant $S_0 > 0$ and a function $\eta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\lim_{r\to\infty} \eta(r) = 0$ and if $r > S_0$, then for $y \in X_1$ the patch B_r^y determines the patch $B_{r-S_0}^{\psi(y)}$ up to motion by some $g \in \mathcal{G}_E$ with $\ell(g) \le \eta(r)$.*

Proof. By uniform continuity of ψ, there is a radius S_0 such that the patch $B_{S_0}^y$ determines the tile at the origin of $\psi(y)$ and its immediate neighbors, up to motion by less than $m/2$. Since ψ is a conjugacy, for any point $\mathbf{a} \in \mathbb{E}^d$, the patch $B_{S_0}(\mathbf{a})^y$ determines the tile at $\mathbf{a}$ in $\psi(y)$ and its nearest neighbors, up to a small motion. Applying this to all points $\mathbf{a} \in B_{r-S_0}$, we have that the patch B_r^y determines $B_{r-S_0}^{\psi(y)}$ up to an overall small motion g. The bound on $\ell(g)$ follows from uniform continuity of ψ. $\square$

For $n \in \mathbb{N}$ put

$$k(n) = \left\lfloor n \frac{\log \lambda_1}{\log \lambda_2} \right\rfloor \tag{16}$$

and set

$$\psi_n = \phi_2^{-k(n)} \circ \psi \circ \phi_1^n. \tag{17}$$

Fix $x \in X_1$. Note that $\lambda_1^n / \lambda_2^{k(n)} \in [1, \lambda_2)$, and if $\log \lambda_1 / \log \lambda_2$ is rational then the sequence $(\lambda_1^n \lambda_2^{-k(n)})_{n=1}^{\infty}$ is periodic and 1 is in its range, while if $\log \lambda_1 / \log \lambda_2$ is irrational then the range of this sequence is dense in $[1, \lambda_2]$. Thus there exists a subsequence n_i such that $\psi_{n_i}(x)$ converges, say to $x' \in X_2$, and such that

$$\lambda_1^{n_i} \lambda_2^{-k(n_i)} \to 1 \qquad \text{as } i \to \infty. \tag{18}$$

Proposition 2. *The sequence* $\{\psi_{n_i}\}_{i=1}^{\infty}$ *converges uniformly to a conjugacy* $\psi' : (X_1, \mathcal{G}_E) \mapsto (X_2, \mathcal{G}_E)$ *such that, for* $r > \rho + S_0$ *and for* $y \in X_1$ *the patch* B_r^y *determines the patch* $B_{r/2}^{\psi'(y)}$.

Proof. Step 1. If $g \in \mathcal{G}_E$ then $\psi_{n_i}(gx) \to gx'$ as $i \to \infty$. Indeed, if we write $gx = \alpha x + \mathbf{s}$ with $\alpha \in SO(d)$ and $\mathbf{s} \in \mathbb{R}^d$ then

$$\begin{aligned} \psi_{n_i}(gx) &= (\phi_2^{-k(n_i)} \circ \psi)(\phi_1^{n_i}(gx)) \\ &= (\phi_2^{-k(n_i)} \circ \psi)(\alpha \phi_1^{n_i}(x) + \lambda_1^{n_i} \mathbf{s}) \\ &= \phi_2^{-k(n_i)}(\alpha \psi[\phi_1^{n_i}(x)] + \lambda_1^{n_i} \mathbf{s}) \\ &= \alpha \psi_{n_i}(x) + \lambda_2^{-k(n_i)} \lambda_1^{n_i} \mathbf{s}, \end{aligned} \tag{19}$$

which clearly converges to gx' as $i \to \infty$.

Step 2. Suppose $g, g' \in \mathcal{G}_E$ and $r > \rho + S_0$ are such that $B_r^{gx} = B_r^{g'x}$. For each i,

$$B_{\lambda_1^{n_i} r}^{\phi_1^{n_i}(gx)} = B_{\lambda_1^{n_i} r}^{\phi_1^{n_i}(g'x)}. \tag{20}$$

By Lemma 10, there exists $h_i \in \mathcal{G}_E$ with $\ell(h_i) \le \eta(\lambda_1^{n_i} r)$ such that

$$B_{\lambda_1^{n_i} r - S_0}^{h_i \psi[\phi_1^{n_i}(gx)]} = B_{\lambda_1^{n_i} r - S_0}^{\psi[\phi_1^{n_i}(g'x)]}. \tag{21}$$

By Lemma 9

$$B_{(\lambda_1^{n_i} r - S_0)/2\lambda_2^{k(n_i)}}^{\phi_2^{-k(n_i)}(h_i \psi[\phi_1^{n_i}(gx)])} = B_{(\lambda_1^{n_i} r - S_0)/2\lambda_2^{k(n_i)}}^{\phi_2^{-k(n_i)}(\psi[\phi_1^{n_i}(g'x)])}. \tag{22}$$

Let $\alpha, \beta_i \in SO(d)$ and $\mathbf{s}, \mathbf{t}_i \in \mathbb{R}^d$ be such that $g\mathbf{a} = \alpha \mathbf{a} + \mathbf{s}$ and $h_i \mathbf{a} = \beta_i \mathbf{a} + \mathbf{t}_i$ for all $\mathbf{a} \in \mathbb{E}^d$. We have

$$\phi_2^{-k(n_i)}(h_i \psi[\phi_1^{n_i}(gx)]) = \beta_i \alpha \psi_{n_i}(x) + \lambda_2^{-k(n_i)} \lambda_1^{n_i} \beta_i \mathbf{s} + \lambda_2^{-k(n_i)} \mathbf{t}_i, \tag{23}$$

and this converges to gx' as $i \to \infty$. We know from Step 1 that

$$\phi_2^{-k(n_i)}(\psi[\phi_1^{n_i}(g'x)]) \to g'x' \text{ as } i \to \infty. \tag{24}$$

Since

$$\frac{\lambda_1^{n_i} r - S_0}{2\lambda_2^{k(n_i)}} \to \frac{r}{2} \qquad \text{as } i \to \infty, \tag{25}$$

it follows that

$$B_{r/2}^{gx'} = B_{r/2}^{g'x'}. \tag{26}$$

Step 3. Define $\psi' : X_1 \mapsto X_2$ as follows. Given $y \in X_1$ let $\{g_j\}_j$ be a sequence in $\mathcal{G}_E$ such that $g_j x \mapsto y$ as $j \to \infty$. There exist group elements h_j tending to the identity in $\mathcal{G}_E$ and positive real numbers r_j tending to infinity such that $B_{r_j}^{h_j g_j x} = B_{r_j}^{y}$ for each j. By Step 2, $(h_j g_j x')_j$ converges, hence so does $(g_j x')_j$, and we define $\psi'(y) = \lim_{j\to\infty} g_j x$. The existence of this limit ensures that ψ' is well defined and continuous. We see from Step 1 that it is a conjugacy, and the sliding block code follows from Step 2.

Step 4. We still need to show that ψ_{n_i} converges to ψ' uniformly on X_1. For $y \in X_1$ and $r > 0$ one can find $g \in \mathcal{G}_E$ such that gx agrees with y on a ball of radius r about the origin. By linear repetitivity (see Lemma 2), we can always choose g such that $\ell(g) < Cr$ for some fixed constant C. By the triangle inequality,

$$\begin{aligned} m[\psi_{n_i}(y), \psi'(y)] \le{}& m[\psi_{n_i}(y), \psi_{n_i}(gx)] + m[\psi_{n_i}(gx), g\psi_{n_i}(x)] \\ &+ m[g\psi_{n_i}(x), \psi'(gx)] + m[\psi'(gx), \psi'(y)]. \end{aligned} \tag{27}$$

Given ϵ, we will show that for i large (with estimates independent of y), and for the correct choice of g, each term on the right hand side is bounded by $\epsilon/4$. The left hand side (which is independent of the choices made) is then bounded by ϵ. Since the estimates on i were independent of y, the left hand side goes to zero as $i \to \infty$ at a rate that is independent of y.

The argument of Step 2 shows that the maps ψ_{n_i} are uniformly continuous with estimates that are independent of i. As a result, the first term can be made small, independent of i (and y), by choosing r large enough. The last term is also small if r is large, since ψ' is uniformly continuous. For fixed r (and hence fixed g), the second term is bounded by $Cr|1 - \lambda_1^{n_i}\lambda_2^{-k(n_i)}|$, which is small if i is large enough. Finally, the third term is small once i is big enough that $\psi_{n_i}(x)$ and $\psi'(x)$ agree up to a small motion on a ball of radius $\gg Cr$ about the origin. □

Proposition 3. *For all sufficiently large i we have $\lambda_1^{n_i}\lambda_2^{-k(n_i)} = 1$.*

Proposition 4. *There exists $I \in \mathbb{N}$ such that for all $i \ge I$, for all $y \in X_1$,*

$$\psi_{n_i}(y) = \psi'(y) + \mathbf{s}_{y,i} \tag{28}$$

for some $\mathbf{s}_{y,i} \in \mathbb{R}^d$.

Proof of Propositions 3 and 4. For fixed $r, \varepsilon > 0$ to be specified in the proof one can find $\delta > 0$ such that if $y, y' \in X_1$ with $d(\psi_{n_i}(y), \psi'(y)) < \delta$ then there exists $g_{y,i} \in \mathcal{G}_E$

with $\ell(g_{y,i}) < \varepsilon$ such that $\psi_{n_i}(y)$ and $g_{y,i}\psi'(y)$ agree on the ball of radius r centered at the origin. Choose I such that $d(\psi_{n_i}(y), \psi'(y)) < \delta$ for all $i \geq I$, $y \in X_1$. Let us consider $i \geq I$. If r is chosen large enough and ε small enough, then $g_{y,i}$ is uniquely determined by the above conditions and varies continuously with y, for otherwise we would have a large patch with a small period, contradicting the recognizability hypotheses (see the remarks following the proof of Theorem 2). Let $\alpha_{y,i} \in SO(d)$ and $\mathbf{s}_{y,i} \in \mathbb{R}^d$ denote the rotational and translational parts of $g_{y,i}$, respectively. We have, for $\mathbf{t} \in \mathbb{R}^d$ with $\|\mathbf{t}\| < r/\lambda_2$,

$$\psi_{n_i}(y + \mathbf{t}) = \psi_{n_i}(y) + \lambda_1^{n_i}\lambda_2^{-k(n_i)}\mathbf{t}, \tag{29}$$

and this agrees with

$$g_{y,i}\psi'(y) + \lambda_1^{n_i}\lambda_2^{-k(n_i)}\mathbf{t} = g_{y,i}\psi'(y + \mathbf{t}) + \lambda_1^{n_i}\lambda_2^{-k(n_i)}\mathbf{t} - \alpha_{y,i}\mathbf{t} \tag{30}$$

on the ball of radius $r - \lambda_2\mathbf{t}$ about the origin. If r is large enough and ε small enough then this implies, for all $\|\mathbf{t}\|$ sufficiently small, for all $y \in X_1$,

$$\alpha_{y+\mathbf{t},i} = \alpha_{y,i} \tag{31}$$

and

$$\mathbf{s}_{y+\mathbf{t},i} = \mathbf{s}_{y,i} + \lambda_1^{n_i}\lambda_2^{-k(n_i)}\mathbf{t} - \alpha_{y,i}\mathbf{t}. \tag{32}$$

By continuity $\alpha_{y+\mathbf{t},i} = \alpha_{y,i}$ for all $\mathbf{t}$, and this implies that the above formula for $\mathbf{s}_{y+\mathbf{t},i}$ holds for all $\mathbf{t}$ as well. Now $\mathbf{s}_{y+\mathbf{t},i} < \varepsilon$ for all $\mathbf{t}$, and this is only possible if $\alpha_{y,i}$ is the identity and $\lambda_1^{n_i}\lambda_2^{-k(n_i)} = 1$. Thus $\psi_{n_i}(y + \mathbf{t}) = \psi'(y) + \mathbf{s}_{y,i} + \mathbf{t}$. □

Corollary 2. *For each $i \geq I$ and each $y \in X_1$, there exists $g_{y,i}$ in the center of $\mathcal{G}_{rel}(y)$ such that $\psi_{n_i}(y) = g_{y,i}\psi'(y)$. Furthermore, if y' is in the closure of the translation orbit of y, then $g_{y',i} = g_{y,i}$.*

Proof. Fix i and y. By Proposition 4 there exists a translation $g_{y,i}$ such that $\psi_{n_i}(y) = g_{y,i}\psi'(y)$. Since ψ_{n_i} and ψ' are conjugacies and all translations commute, we have $g_{y,i} = g_{y',i}$ for any y' in the translation orbit of y. By continuity, this last equality holds for all y' in the closure of the translation orbit of y.

To show that $g_{y,i}$ is in the center of $\mathcal{G}_{rel}(y)$, it suffices to show that $\alpha g_{y,i} = g_{y,i}$ for every $\alpha \in SO(d) \cap \mathcal{G}_{rel}(y)$. Fix such α. By definition of $\mathcal{G}_{rel}(y)$ there is a sequence $h_j \in \mathcal{G}_{rel}(y)$ such that $h_j y \to y$ as $j \to \infty$ and such that the rotational part α_j of h_j converges to α. If $g_{y,i}$ is translation by $\mathbf{s}_{y,i}$ then

$$\begin{aligned}
\psi'(y) + \mathbf{s}_{y,i} &= \psi_{n_i}(y) \\
&= \lim_{j\to\infty} \psi_{n_i}(h_j y) \\
&= \lim_{j\to\infty} h_j\psi_{n_i}(y) \\
&= \lim_{j\to\infty} h_j g_{y,i}\psi'(y) \\
&= \lim_{j\to\infty} h_j\psi'(y) + \alpha_j\mathbf{s}_{y,i} \\
&= \lim_{j\to\infty} \psi'(h_j y) + \alpha_j\mathbf{s}_{y,i}, \\
&= \psi'(y) + \lim_{j\to\infty} \alpha_j\mathbf{s}_{y,i},
\end{aligned} \tag{33}$$

and hence $\alpha_j\mathbf{s}_{y,i} \to \mathbf{s}_{y,i}$. It follows that $g_{y,i}$ commutes with α. □

Proposition 5. *If two tilings y, y' in a substitution tiling space X_ϕ agree on a single tile, then they are in the same translation orbit closure.*

Proof. Suppose that $y, y' \in X_\phi$ are tilings in different translation orbit closures which agree on a tile; without loss of generality, we may assume that the interior of the tile contains the origin, and thus, $d(\phi^n(y), \phi^n(y')) \to 0$ as $n \to \infty$.

There exists a tiling z in the translation orbit closure of y and a rotation $\alpha \in SO(d)$ such that $\phi(z) = \alpha z$. Let $\{n_i\}$ be an increasing sequence of integers such that α^{n_i} converges to the identity. Let $\beta \in SO(d)$ be such that βz is in the translation orbit closure of y'. Then, for each $n_i \geq 1$, $\alpha^{n_i} y$ is in the translation orbit closure of $\phi^{n_i}(y)$, and $\phi^{n_i}(\beta z) = \beta \alpha^{n_i} z$ is in the translation orbit closure of $\phi^{n_i}(y')$. Taking a limit as $n_i \to \infty$, it follows that the distance from the translation orbit closure of y to that of βz is zero, hence that y and βz, and therefore y', are in the same translation orbit closure. □

Remark. Using property iii), one can take α to be the identity, thereby simplifying the proof of Proposition 5. The above argument, however, shows that the conclusions of Proposition 5 apply even to substitutions that do not have a fixed point.

Proof of Theorem 1. First we show that ψ_{n_i} is a sliding block code for $i \geq I$. Suppose $x, y \in X_1$ agree on a large ball around the origin, so that $\psi'(x)$ and $\psi'(y)$ agree on a (smaller) ball around the origin. By Proposition 5, x and y lie in the same translation orbit closure. However, ψ_{n_1} and ψ' differ by a (fixed) translation on this orbit closure, so $\psi_{n_i}(x)$ and $\psi_{n_i}(y)$ agree on a (still smaller) ball around the origin.

Now note that $\psi = \phi_2^{k(n_i)} \psi_{n_i} \phi_1^{-n_i}$ is a composition of three maps, each of which is a sliding block code up to scaling. Each patch in $\psi(x)$ is determined by a (much smaller) patch in $\psi_{n_i} \circ \phi_1^{-n_i}(x)$, which is determined by a patch in $\phi_1^{-n_i}(x)$, which is determined by a (much larger) patch in x. Thus ψ is a sliding block code. □

Corollary 3. *If the translation orbit of a tiling is dense in X_1, then ψ intertwines the actions of some powers of ϕ_1 and ϕ_2.*

Proof. In this case $\mathcal{G}_{rel}(y)$ of a tiling y has no center, so g_i is the identity and $\psi_{n_i} = \psi'$ for all $i \geq I$. But then

$$\phi_2^{-k(n_i)} \circ \psi \circ \phi_1^{n_i} = \psi_{n_i} = \psi' = \psi_{n_{i+1}} = \phi_2^{-k(n_{i+1})} \circ \psi \circ \phi_1^{n_{i+1}}. \tag{34}$$

Multiplying on the left by $\phi_2^{k(n_{i+1})}$ and on the right by $\phi_1^{-n_i}$ gives

$$\phi_2^{k(n_{i+1})-k(n_i)} \circ \psi = \psi \circ \phi_1^{n_{i+1}-n_i}. \tag{35}$$

□

3. Proof of Theorem 3

Suppose that $\psi : X \to Y$ is a topological conjugacy and that Y is of finite type. By assumption, there exists a finite length r_1 such that $Y_{r_1} = Y$.

As in the proof of Theorems 1 and 2, let D be a length greater than the diameter of any tile in either tiling system. By finite local complexity there exists a radius $m > 0$ such that every tile contains a ball of radius m and the only way to move two adjacent tiles a distance m or less and obtain an admissible local pattern is to move the pair by a rigid motion. Since ψ^{-1} is uniformly continuous, there is a length r_2 such that, for each $y \in Y$, $B^y_{r_2}$ determines $B^{\psi^{-1}(y)}_{2D}$, up to a Euclidean motion that moves each point in B_{2D} by a distance of $m/2$ or less. Likewise, there is a length r_3 such that, for each $x \in X$, $B^x_{r_3}$ determines $B^{\psi(x)}_{r_1+r_2+3D}$ up to a wiggle of size at most $m/2$.

We claim that $X_{r_3} = X$, and thus that X is of finite type. For if $x \in X_{r_3}$, then every patch of radius r_3 in x corresponds to an admissible patch in a tiling in X, and so determines a patch of a tiling in Y (up to a small motion). That is, the tiling x determines a combinatorial tiling of the tiles of the Y system, such that each patch of radius $r_1 + r_2 + 3D$ is actually admissible. This local information can be stitched together to form an actual tiling $y \in Y_{r_1+r_2+3D} = Y$. The tiling $\psi^{-1}(y)$ is then a tiling in X. However, the combinatorial structures of x and $\psi^{-1}(y)$ are the same, since $B^x_{r_3}(\mathbf{a})$ determines $B^y_{r_1+r_2+3D}$ up to a small rigid motion, which determines $B^{\psi^{-1}(y)}_{2D}$ up to small rigid motion. Since the tiles are rigid, this implies that x is obtained by applying a rigid motion to $\psi^{-1}(y)$, and is thus in X. □

4. Conclusions

We have been concerned with topological conjugacy between tiling dynamical systems, emphasizing the geometric aspects by including systems in which the tiles appear, in each tiling, in infinitely many orientations, thus incorporating the rotation group in an essential way. Some of our results are restricted to a subclass of tiling dynamical systems, substitution systems, which can be thought of as incorporating an extra group action which represents a certain similarity: that is, not only have we extended the usual action of the translation group by the rotations, we have in fact extended further, to a subgroup of the conformal group.

Our first result is to show that topological conjugacies between substitution systems with invertible substitutions are quite rigid. We show that any conjugacy for the Euclidean actions automatically extends to (some powers of) the similarities, and can be represented by the natural analogue of a sliding block code. Tiling dynamical systems are a geometric extension of subshifts, and these results all have significant geometric meaning.

Our second result is that substitutions whose systems do not admit periodic tilings are recognizable, as long as the relative orientation group is either finite [S] or dense in $SO(d)$. In particular, all nonperiodic substitutions in two dimensions are recognizable. This result can then be used to generalize constructions such as those in [PS] from the category of translationally finite tilings to the more general case where tiles can appear in arbitrary orientation.

Part of the significance of substitution subshifts and substitution tiling systems for dimension $d \geq 2$ arises from the fact that, quite generally (see [Moz], [G]), such a system carries a unique invariant measure and is measurably conjugate to some uniquely ergodic system of finite type. Actually, the proofs show more than measurable conjugacy; they show that off sets of measure zero the map is bicontinuous, and it is boundedly finite to one on the sets of measure zero. These associated finite type systems are also of geometric interest, as part of the general effort of understanding the symmetries of densest packings of bodies [R2]. Our third result is a step in this direction, showing that finite type is a topological property among tilings with finite local complexity, and not merely an artifact of the way one defines the tiles.

It would be significant if Theorem 1, and the conjugacy invariants (the relative orientation groups) which follow from it, apply to such tiling dynamical systems. For instance, we noted above that the (1,2)-pinwheel and (3,4)-pinwheel systems cannot be topologically conjugate. We thus conclude with an open problem.

Open Problem. Are the two finite type tiling systems, which are measurably conjugate to the (1,2)-pinwheel and (3,4)-pinwheel systems, topologically conjugate?

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